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This report presents a more detailed elaboration and discussion of the mathematical content of the central concepts which have been designated as the Mathematical Strands of the Mathematics Curriculum for grades K-8. In Part I of the report there is a discussion of the pedagogical philosophy which the committee felt was to have an importance equal to that of the mathematical content itself. In Part II the implications of that philosophy are readily apparent in the development of "strands," the basic cognitive subdivisions of the mathematics curriculum. Within each strand are listed examples to convey the general scope and sequential development of the mathematical content. The strands which have been selected are divided into two categories. The first category includes the strands of number and operation, geometry, measurement, and probability and statistics. The other category includes the strands of applications, sets, functions, logical thinking, and problem solving. (RP)

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MATHEMATICS PROGRAM, K-8

1967-1968 STRANDS REPORT

PART 2

BY

Statewide Mathematics Advisory Committee

PRELIMINARY

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CALIFORNIA STATE DEPARTMENT OF EDUCATION
Max Rafferty—Superintendent of Public Instruction
Sacramento 1968

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Statewide Mathematics Advisory
Committee

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STATEWIDE MATHEMATICS ADVISORY COMMITTEE

The names of the members of the Statewide Mathematics Advisory Committee and the positions they held at the time of the study follow:

John L. Kelley, Chairman, Professor of Mathematics, University of California, Berkeley

*G. Don Alkire, Professor of Mathematics, Fresno State College

George F. Arbogast, Elementary Mathematics Supervisor, Los Angeles City Unified School District

Leslie S. Beatty, Director of Research and Curriculum Materials, Chula Vista City Elementary School District

William G. Chinn, Executive Director, SMAC Research Project, San Francisco City Unified School District

Richard A. Dean, Professor of Mathematics, California Institute of Technology, Pasadena

Donald D. Hankins, Mathematics Specialist, San Diego City Unified School District

Robert A. Hansen, Administrative Assistant to the Superintendent, Fresno City Unified School District

Albert Hanson, Chairman, Mathematics Department, Carlmont High School, Redwood City

Holland I. Payne, Program Specialist in Mathematics, Sacramento City Unified School District

Bryne Bessie Frank, Executive Secretary, Consultant in Mathematics, Bureau of Elementary and Secondary Education, State Department of Education

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* G. Don Alkire resigned on July 18, 1967. The State Board of Education appointed his replacement, Robert A. Hansen.

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PART II

This part of our report presents a more detailed elaboration and discussion of the mathematical content of the central concepts we have designated as the Mathematical Strands of the mathematics curriculum for grades K-8.

In the first part of our report we discussed the pedagogical philosophy which we believe to have an importance equal to that of the mathematical content itself. The implications and overtones of that philosophy are readily apparent in the development of each strand.

We are hopeful that this document will be of use to writers, publishers and teachers. We do not seek to be prescriptive to authors and publishers, nor do we intend this as an outline for an in-service course for teachers. Within each strand we list examples to convey the general scope and sequential development of the mathematical content. We hope that writers, publishers and teachers will find here a strong and clear statement of many subtle and crucial issues in curriculum planning together with our recommendations and commitments on these issues. The obvious restrictions of space and time have not permitted us to give a definitive treatment of any one strand.

The strands which we have selected for special emphasis split naturally into two categories. The first category includes the Strands of Number and Operation, Geometry, Measurement, and Probability and Statistics. These strands are the basic cognitive subdivisions of the mathematics curriculum itself. The other category includes the Strands of Applications, Sets, Functions, Logical Thinking, and Problem Solving. These strands are catalysts, or processes, which are present to some degree in every mathematical enterprise to facilitate mathematical analysis. No single Strand can stand by itself; together they constitute a strong viable program. To slight one Strand is to significantly weaken this program. It would be equally unfortunate to give undue emphasis to one particular Strand. A satisfactory curriculum will display and use these interdependent strengths in its development.

A word about algebra. We believe that the next five years will see more and more students ready for a first course in algebra at the beginning of Grade 8. Indeed this is a conservative prediction if the program we envisage has the merit we claim for it and if the program is adopted and implemented with a significant in-service training program. We can even anticipate the day when every college capable student will have had algebra before entering Grade 9.

We do not feel that algebra should be identified as a strand in itself, anymore than arithmetic should be considered a strand. The central ideas of a modern algebra course can easily be identified as an extension of the framework of the Strands described here. We have not chosen to show this extension in each Strand because we feel that the major emphasis of our recommendations must concern the mathematical training of students up to algebra. On the other hand we do feel that algebra is appropriate for Grade 8 and we recommend that it be included for as many pupils as are now prepared for it. For this reason we have added, as a coda, a brief outline of what we envisage as a suitable program for algebra. This program is based upon our recommendations for what we have called the K-8 curriculum.

Strand 1. Numbers and Operations

The major content of the study of numbers in the mathematics program for Kindergarten through Grade Eight is a full development of the rational number system and an introduction to the real number system.

What is a number? In the discipline of mathematics, a number is an abstract object identified by its properties. However, in the instructional program in mathematics, numbers must be regarded as ideas derived from and applicable to situations and problems encountered by the learner. The program must provide activities which guide pupils from intuitive recognition of relationships through increasingly systematic ways of thinking. In developing ability to work with abstract symbols, most learners will need to "look through" symbols to the images of events in their experience which have given meaning to the symbols. These images may be of a physical activity in which a relation was recognized, a picture or a diagram which served to clarify a relation, or a pattern of symbols of a lower order of abstraction.

The system of rational numbers should be presented as a system of expanding ideas. In such a presentation the learner encounters the whole numbers $\{0, 1, 2, \ldots\}$. These numbers are ordered, named by a positional system of notation, and the basic operations with their properties are introduced. The system of positive rational numbers and 0 (sometimes called the fractional numbers) can be studied next. The system of whole numbers is extended to the integers by considering the numbers $-1, -2, -3, \ldots$. Finally, the study of such numbers as $-\frac{1}{4}$, $-\frac{7}{8}$, $-1\frac{1}{2}$ completes the system of rational numbers. The operations and properties, first developed for the whole numbers, are extended to each system as the study proceeds.

Throughout this development the number line should be employed to provide a geometric representation of number leading to the idea that each rational number corresponds to a point on the number line. By Grade 8 it will be known that, for example, $\sqrt{2}$ is not a rational number, yet by a geometric construction there is a point on the number line which, in a natural way, should correspond to $\sqrt{2}$. The real number system may then be developed as a completion of the rational numbers on the number line so that every point of the number line is in one-to-one correspondence with a number.

While the sequential development should in general be carried out as described above, there must be considerable overlapping. Preparatory activities for more formal study of each extension of the number system begin early and should continue through a long period of time. For example, experiences with fractional parts of objects may occur in kindergarten, several years before the learner works with rational numbers as such. At the upper level, preparation for study of the real numbers includes the decimal expansion of numbers. The decimal expansion of a rational number is periodic, e.g., .7143143 . . ., .5333 . . ., or .25000, and conversely a periodic decimal names a rational number.

There are comparatively few difficulties in developing the concepts and computational algorithms for the system of whole numbers. The development of the full system of rational numbers requires two stages. One stage is the introduction of the positive rationals: "How can you divide 1 by 2?" "Why doesn't $\frac{2}{3} + \frac{3}{4} = \frac{2}{4}$?" The other stage is the introduction of negative numbers: "How can you subtract 3 from 1?" "Why doesn't $-\frac{7}{8} = \frac{-7}{-8}$?" It is logically immaterial whether the introduction of negative numbers precedes or follows the introduction of the positive rational numbers. Both stages are analogous and yet each contains its own subtleties.

Pedagogically, we recommend a course which deals independently with these stages. This is particularly important in the early grades. There are many instances when it is natural to introduce a few of the negative numbers, say $\{-10, -9, -8, \ldots, -1\}$, as a countdown procedure, or to label points on a temperature scale or on the other "half" of the number line. Easy arithmetic problems involving addition and subtraction of small positive and negative integers can be handled. For example: If the spacecraft was on automatic control from H-2 hours until H + 1 hours, how long was it on automatic control?

At the same time children will often need to use the familiar positive rational numbers, $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\}$ to express parts of a whole. These should be introduced informally as the need arises and experiences should be tied closely with physical problems. Here again it is possible to handle easy situations of addition and subtraction without the complicated fanfare of the most general of rules. The more difficult arithmetic operations of multiplication and division and the general rules for these as well as for addition and subtraction in the full set of rational numbers occur later in the curriculum. These topics require a fuller understanding of the number system.

Figure 1 shows the relationship of the whole numbers, the integers, the rational numbers, and the real numbers.

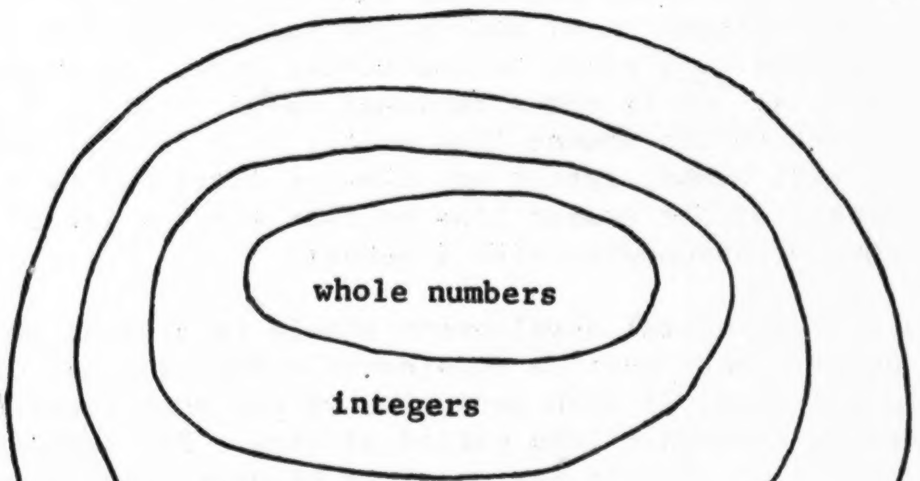


Figure 1

Activities that guide the learner to recognize and generalize the central unifying ideas in these number systems aid him to progress toward systematic thinking. Generalizations feed back, giving support to existing ideas, as well as feed forward, opening the way for new ideas.

Early informal experiences which provide intuitive background for the development of systematic ideas at a later stage are difficult to categorize. In kindergarten and early primary grades activities such as those in which a child builds with blocks and other materials, handles objects of different shapes and notes characteristic features, sorts and classifies objects in order of size, fits objects inside others, experiments with a balance, recognizes positional relationships, recognizes symmetry, searches for patterns, and reasons out ideas are highly important. Through such activities children develop ideas of relationships that contribute to the Strand of Numbers and Operations as well as to other strands of mathematics.

Situations that occur naturally in the classroom or on the playground, as well as activities with materials specifically designed to promote development of certain ideas, games, and puzzles provide experiences of the type described above. While engaged in these activities, children should be encouraged to talk about what they are doing, both with the teacher and with each other. These articulations provide a readiness for reading and writing mathematics.

The teacher introduces new words or language patterns in close association with activities that make meanings apparent. Assimilation of vocabulary and language patterns into the children's own speech and thought is expected to occur gradually over a considerable period of time. Children first give evidence of understanding the teacher's usage; then they begin to use the words and patterns spontaneously in their speech. Often the first uses are imprecise. The teacher accepts and perhaps even repeats a child's speech. Then, sensitive to his intended meaning, the teacher guides him to precision through a skillful question, a statement of agreement that rephrases the idea, or a statement of amplification. Development of ideas and language can be blocked by rejection of a child's first fumbling attempts to cope with what may be, to him, enormous complexities.

The principal unifying ideas in the strand of Number and Operation are:

Order, Counting, and Betweenness
Operations and their Properties
Identity Elements
Numeration Systems
Mathematical Sentences

Order, Counting, and Betweenness.

For any pair of numbers, a, b, exactly one of three possible relations exists:

$$a < b, \qquad a = b, \qquad \text{or} \qquad a > b.$$

Activities in which children compare the numbers of sets of objects without resorting to counting lead directly to the concept of the counting numbers. Learners can compare the numbers of two sets of objects by pairing the members one-to-one and discover the possible relations. For example, if chairs and children are paired to see whether there is a chair for each child, one of the relations more, fewer, or as many as, will be found to exist.

Once a child can compare sets of objects and determine whether they are equivalent or non-equivalent, the learning of relations between numbers ($<, =, >$) can be promoted by arranging sets of objects in sequential order. By this time, many children have learned to associate the spoken names of numbers with the cardinal numbers of small sets of objects and zero with the number of the empty set. Experiences in arranging sets of objects in rank order help pupils to see the order of numbers and to learn the names of the natural numbers in counting sequence.

Counting requires matching the members of a set of objects with the members of an ordered set of names of the counting numbers. Some children require help in keeping the ordered set of number names in one-to-one correspondence with a set of objects they are counting, and must move, touch, or point to the objects and say the words aloud if they are to count correctly. Many activities which involve counting in a variety of situations are important to avoid confusions arising from partial learnings.

The number line, which relates geometry and number, exhibits many important facts about numbers. In early experiences, a number line can be marked or laid out on the floor with masking tape. A starting point (origin) can be identified and the children can take steps along the line, counting as they step. Soon numerals may be associated with the points, with 0 as the origin. Numbers of steps taken along the line by different children can be compared and relative positions discussed, for example, before, between, after. Later, movement along the line may be represented by marking a number line drawn on the chalkboard or paper. Such activities provide intuitive background for ideas that will not be formulated in words until long after these initial experiences.

When the number line is used, $a < b$ means that the point corresponding to a is located to the left of (or below) the point corresponding to b. A number (point) a is said to lie between two other numbers (points), b, c if $b < a$ and $a < c$. For any three distinct numbers, x, y, z, exactly one of the three lies between the other two.

When the fractional representation for rational numbers is studied, the number line helps pupils see that numbers whose fractions have the same denominator are ordered by the numerators of the fractions. They find that they can make a decision concerning the order of any two rational numbers by renaming them so that the fractions have the same denominators.

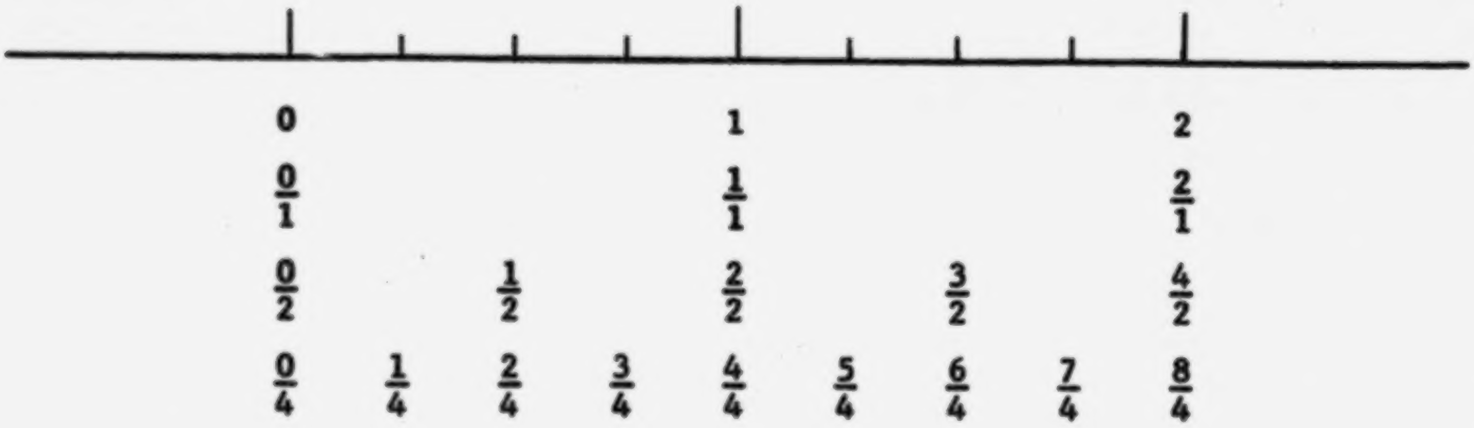


Figure 2

The number line shows that any negative number is to the left of (or below) zero and is hence less than zero, and -4 is less than -1.

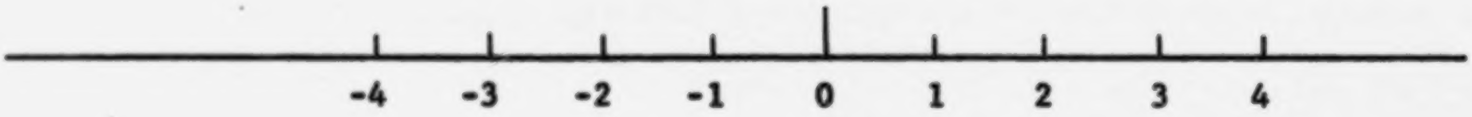


Figure 3

When using both positive and negative numbers it is often useful to employ the concept of absolute value. The absolute value of the number a, denoted $|a|$, can be thought of as the distance from the point corresponding to the number a to the origin. It may be defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

The number line should be used to point up the distinction between the set of integers and the set of rational numbers. The rational numbers are dense; between any two distinct rational numbers there is always another rational number. Thus between a and b is the number $\frac{a+b}{2}$.

The number plane should also be introduced at an early stage. Here points in a plane are identified by an ordered pair of numbers. The number plane should be related to map drawing. The number plane lends itself to interesting games and the study of patterns. For example, how can the points be described which lie on the line through (0,0) and (1,2)? The number plane is necessary for an understanding of functions and graphing.

Operations and their Properties.

Addition is an operation which assigns to an ordered pair of numbers, called addends, another number called their sum. For example, to the ordered pair of numbers (3,4) is assigned 7, their sum. The fact that the ordered pair (4,3) is also assigned the sum 7 is an example of the commutative property of addition. On the set of integers subtraction is also an operation. Subtraction is the inverse operation of addition. If the sum and one addend are known, the other addend may be found by subtraction. Multiplication is an operation which assigns to an ordered pair of numbers, called factors, another number called their product. For example, to the ordered pair of numbers (3,4) is assigned 12, their product. Division is the inverse operation of multiplication. If the product and one factor are known the other factor may be found by division.

Concepts of the operations and properties have their roots in activities that involve the joining and separation of nonintersecting sets of objects. For example, suppose Terri and Charles are painting at easels and Sara asks to paint also. The teacher may ask, "How many children are at the easels now? How many will there be when Sara joins them? Let's count to see if we were correct." When the operation of addition can be associated with the numbers of the sets then the number sentence $2 + 1 = 3$ is written. Children now see, once again, that "2 + 1" and "3" are names for the same number.

In experiences leading to subtraction the learner separates a set into two disjoint subsets and names the numbers. He finds that if he knows the number of the original set and one of the subsets, he can find the number of the other subset by subtraction. It should be noted that the learner is not expected either to follow or to give a wordy description of the relationship, which should be made clear through activities with such materials as sets of objects, scaled blocks or paper strips, and the number line. Pupils should experience various interpretations of subtraction including "take away", "comparison", "how many more", and the inverse relation of subtraction and addition.

Either addition or subtraction involves three distinct sets: two disjoint sets, A and B, and their set union, $C = A \cup B$. Any of the following describes the numbers in the sets:

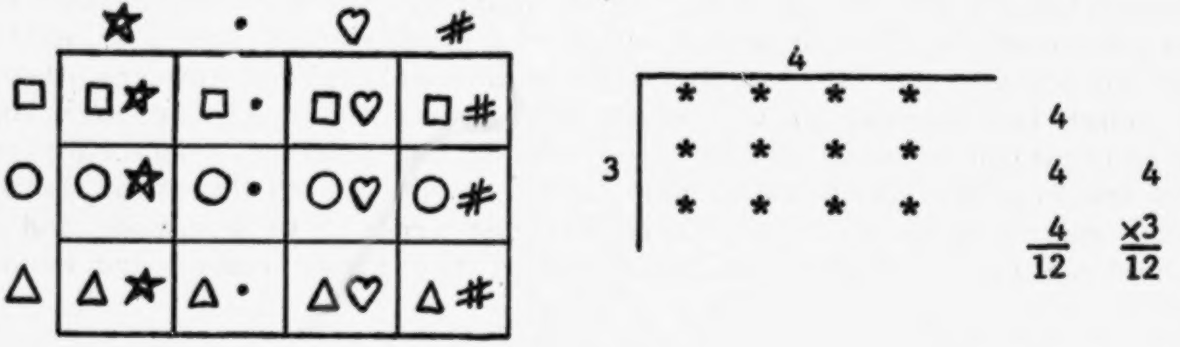
$$N(A) + N(B) = N(C) \qquad N(C) - N(A) = N(B)$$
$$N(B) + N(A) = N(C) \qquad N(C) - N(B) = N(A)$$

Note also that the question, "What number must be added to 3 to equal 7?" has the symbolic form of an equation: $3 + \square = 7$. The symbol 7-3 names, by definition, a number which when added to 3 is 7.

In the learner's early experiences with multiplication, the operation is defined by example in terms of counting the members of a set formed by joining disjoint sets that are equivalent. A rectangular array of objects is helpful in clarifying this interpretation of the operation. Appeal to the idea of repeated addition of equal addends may aid the learner in interpreting the operation.

The Cartesian product interpretation of multiplication extends the concept and makes it applicable to types of problem situations for which repeated addition is not an adequate model. For example, suppose one wants to find the number of different combinations one can make from three shirts and four pairs of slacks, or the number of different combinations that occur under the operation of addition with the numbers 0 through 9.

In the Cartesian product interpretation of multiplication, we pair the elements of one set with the elements of another and then enumerate the number of ordered pairs that results. Figure 4 illustrates the possible pairings of a set of three objects with a set of four in a rectangular array. Each of the pairings can then be represented more simply. We can determine the number of elements in the array by a variety of methods, -- counting, repeated addition and multiplication.



$4 + 4 + 4 = 12$

$3 \times 4 = 12$

Figure 4

Division is the operation for finding one factor if the product and the other factor are known. In the early stages of learning, the relation is made clear by separating a set into equivalent subsets and determining the number of them. Appeal to the idea of successive subtraction may aid children in interpreting division. The number line can be a helpful aid in illustrating this interpretation of division.

Note that division is the operation which solves, for example, the equation $3 \times \square = 12$. The symbol $12 \div 3$ names, by definition, a number which when multiplied by 3 is 12.

Another important aspect of division is "division with a remainder". Within a set of numbers, for example, the set of positive integers, it is not always possible to divide one number by another. (Indeed, it is this deficiency of the integers that motivates the construction of the rational numbers.) Thus, for example, 14 cannot be divided by 3; there is no integer whose product with

3 is 14; the equation $3 \times \square = 14$ has no solution in the set of integers. We can however write $14 = 3 \times 4 + 2$. An array such as the one in Figure 5 illustrates this relationship.

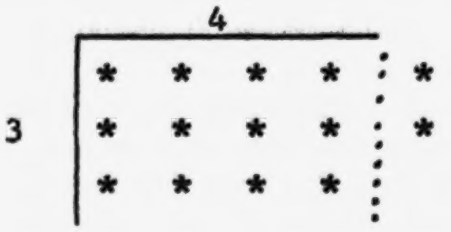
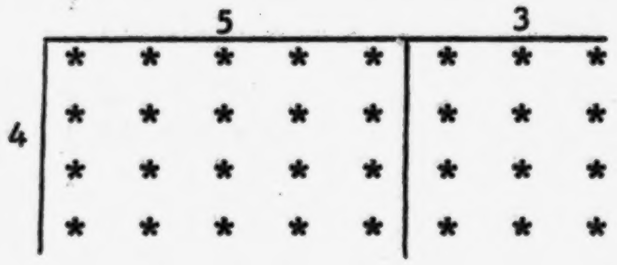


Figure 5

Properties of the operations. Both addition and multiplication are commutative, that is, the order in which two numbers are added or multiplied does not affect the result. Both addition and multiplication are also associative, that is, the way in which the numbers are grouped for addition or for multiplication does not affect the sum or the product. Subtraction and division are neither commutative nor associative. Multiplication is distributive over addition. Figure 6 illustrates that the product of a number and the sum of two numbers is the sum of the products of the first number and each of the addends.



$$4 \times (5 + 3) = 4 \times 8 = 32$$
$$(4 \times 5) + (4 \times 3) = 20 + 12 = 32$$

Figure 6

Distributivity of multiplication over addition and subtraction is an extremely useful and important property, since it is basic to all multiplicative algorithms. It should be emphasized in the elementary school program.

Division distributes over addition and subtraction in the form, for example

$$(8 + 12) \div 4 = (8 \div 4) + (12 \div 4)$$

since $20 \div 4 = 2 + 3$

$5 = 5.$

On the other hand

$$20 \div (2 + 2) \neq (20 \div 2) + (20 \div 2)$$

since $5 \neq 10 + 10.$

The teacher should know and understand mathematical terms, for example commutative, associative, distributive, identity, element, opposite, inverse, absolute value. When the learner gives evidence of familiarity with an idea, the teacher can begin to use a new term easily and informally and give him opportunity to build it into his own speech. Later, attention should be given to it in the reading vocabulary. We suggest that mathematics books can provide an excellent source of material for teaching the reading of mathematical content, and might, on occasion, be included in the reading program. Mathematical experiences are also a rich source of ideas for children's writing.

By the end of the elementary school mathematics program, most pupils should be able to understand and read the standard terminology and language patterns for mathematical concepts which they have learned. Learning to understand and read specialized language takes place through many opportunities for use rather than through memorization and parrot-like repetition.

It is doubtful that either sound knowledge of underlying structure and principles or facility in computation can stand alone. One enhances the other not only in its usefulness but also in its attainment. Skills gained in the absence of understanding are soon forgotten and not readily transferable to different situations, and concepts attained without the support of skills are frequently not operational. We recommend experiences that enable the learner to develop a degree of facility in computation that gives him confidence in his ability to deal with numbers and their applications. Since much of the arithmetic of daily life involves estimating and computing without the use of pencil and paper, these are important aspects of computation.

Sets of exercises arranged so that patterns may be discovered assist in the development of generalizations. When children have stated a general idea, such questions as, "Does this idea always work?", "Does the idea work with larger numbers?", "Can you find an example in which the idea does not work?", stimulate practice that reinforces knowledge of basic facts and develops confidence in working with greater numbers.

Identity Elements.

The numbers zero and one have special properties with respect to addition and multiplication. The addition of 0 to a number results in the same (the identical) number. Thus $A + 0 = 0 + A = A$ for all numbers A. Hence zero is called the identity element with respect to addition.

Similarly one (1) is the identity element with respect to multiplication. This concept plays an important role in computation with fractions. Through experiences with objects, the number line, and geometric figures separated into congruent regions, learners may be guided to see that 1 may be expressed as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3} \ldots$ Then they can discover why $\frac{2}{8}$ is another name for $\frac{1}{4}$, $\frac{6}{9}$ is another name for $\frac{2}{3}$, and so on. For example

$$\frac{1 \times 1}{1 \times 4} = \frac{2 \times 1}{2 \times 4}; \qquad \frac{1 \times 2}{1 \times 3} = \frac{3 \times 2}{3 \times 3}$$

The reason is, of course, that multiplying both the numerator and the denomi- by the same counting number amounts to multiplying the fractional number by 1; $\frac{a}{b} = \frac{k}{k}\,\frac{a}{b}$. Here is a table which may be useful in comparing the properties of the rational numbers under addition and multiplication.

Addition (+)		Multiplication (×)
$a + b = b + a$	Commutativity	$a \times b = b \times a$
$a + (b+c) = (a+b) + c$	Associativity	$a \times (b \times c) = (a \times b) \times c$
0: $0 + a = a + 0 = a$	Identity element	1: $1 \times a = a \times 1 = a$
$a + (-a) = 0$	Inverses	If $a \neq 0$ then $a \times (\frac{1}{a}) = 1$
If $a + b = a + c$ then $b = c$	Cancellation	If $a \neq 0$ and $a \times b = a \times c$ then $b = c$

Distributive Laws

$$a \times (b+c) = (a \times b) + (a \times c)$$

$$(b+c) \times a = (b \times a) + (c \times a)$$

Order

For all a, b exactly one of the alternatives,
$a = b$, $a < b$ or $b < a$, is true.

If $a < b$ then $a + c < b + c$

If $a < b$ and $0 < c$ then $a \times c < b \times c$

Numeration Systems.

Symbols are necessary both for purposes of communication and for efficiency and economy of thought. A numeration system is a set of symbols for naming and recording numbers. A number is a mathematical concept. A numeral is a symbol for a number and names the number. For example, the number eight, the concept which is associated with any set having the property of containing eight members, has infinitely many names. Among them are:

eight, 8, VIII, $5 + 3$, $32 \div 4$, $\frac{24}{3}$, 2^3, and 10_{eight} .

Certain rules or principles govern the use of the symbols in any system of numeration. The two major principles of a place-value system of numeration are base and place. The numeration system which we commonly use is the decimal place-value system. In this system, with its ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, place value is the fundamental principle for organizing the naming of numbers beyond nine. The ten digits can be placed in many positions, arranged according to ordered powers of base ten, to express any number in the system of real numbers.

A learner's concept of the decimal place-value numeration system has its beginning in pre-school or kindergarten experiences when he first begins to name numbers. The concept is refined and extended throughout the entire mathematics curriculum. Among the ideas involved as understanding of decimal place-value notation develops are

- **the positional relationships, left and right**

  concepts of the whole numbers zero through nine and their order
- **symbolizing value by the position, or place, of a digit in a numeral**
- **naming numbers in different ways, for example, naming 532 as**

  5 hundreds, 3 tens, 2 ones
  53 tens, 2 ones
  532 ones
  5 hundreds, 2 tens, 12 ones
  4 hundreds, 13 tens, 2 ones
  4 hundreds, 12 tens, 12 ones
- **the operations of addition and multiplication, for example**

  $532 = (5 \times 100) + (3 \times 10) + (2 \times 1)$ **or**
  $.532 = (5 \times \frac{1}{10}) + (3 \times \frac{1}{100}) + (2 \times \frac{1}{1000})$
- **multiplying and dividing by tens and powers of ten**

  exponential notation, for example

  $532 = (5 \times 10^2) + (3 \times 10^1) + (2 \times 10^0)$
  $.532 = (5 \times 10^{-1}) + (3 \times 10^{-2}) + (2 \times 10^{-3})$

Consideration of systems of numeration which do not utilize the principle of place value can lead to appreciation of the advantages of a place-value system and of the historical development of numeration systems. Study of place-value systems with bases other than ten can contribute to understanding of the principles of base and place in the familiar decimal system. Recognition that properties of numbers are not dependent on a specific numeration system should be a goal of such studies. However, systems of numeration other than the decimal system should not become a major part of the mathematics program.

Mathematical Sentences.

When a learner has had many experiences with joining and separating sets of objects and can recognize and name the numbers of the sets, he can begin

to use number sentences to record and communicate his ideas. He may, for example, find that the union of two disjoint sets, one of four blocks and the other of two blocks, is a set of six blocks. The sentence associated with the numbers of the sets and the operation of addition is, four plus two equals six. Expressed with numerals and signs, the sentence is $4 + 2 = 6$. This form is shorter and easier to write than when words are used. Many experiences with objects, pictures, and the number line lead to learning the basic pattern _____ + _______ = ________. Initially pupils count or simply recognize the numbers of small sets to find the numbers they will name in a sentence. Through such experiences they develop understanding of addition and begin to learn number facts, which enable them to perform the operation more efficiently. Basic sentence patterns for the other operations are established in similar fashion, that is from experiences in "real" situations, to spoken sentences concerning the numbers, to written symbolism.

A number sentence is simply a way of making a statement about numbers, and when it contains only numbers and relational symbols it is either true or false. In the example above, the statement asserted the equality; i.e., the logical identity, of the number named by $4 + 2$ and 6. The statement is true. The statement $4 + 2 = 5$ is false, since $4 + 2$ and 5 do not name the same number. The statements of inequality, $4 + 2 \neq 5$ and $5 < 4 + 2$, are true statements.

In number sentences various symbols, such as _____, □, △, ?, x, y, a, c, k, are used to represent numbers from specified sets. Equations or inequalities that contain such symbols are called open sentences. Their truth or falsity depends upon the numbers used to replace the symbols.

Experiences with sentences having variables in different positions extend an understanding of operations and properties and provide an intuitive background for algebra. Geometrically shaped frames are useful in early experiences, since children can write numerals within them and are not hampered by the necessity of rewriting. However, as soon as children can easily distinguish between numerals and letters, they should have experiences with the conventional use of letters for variables. Solution of number sentences can form the basis for interesting puzzles and games which, in turn, generate fun and learning simultaneously.

Number sentences are particularly useful in guiding learners to generalize relationships and properties of numbers. They provide a form for recording ideas in which patterns may be discovered and for making general statements concerning them. For example, pupils may be challenged to find the "secret" for solving such sentences as $5 + \triangle = 5$ to gain insight into the role of identity elements. Such statements as $(12 + \square) - \square = 12$ give insight into "doing" and "undoing", that is, inverse relations. When the secret has been discovered, a generalized statement may be formulated, $(\square + \triangle) - \triangle = \square$, or $(a + b) - b = a$ for all numbers a and b.

Sentences should increase in complexity as children develop ability and need to analyze more complicated problems. The use of parentheses requires special attention. Probably parentheses will first appear in the associative law: $2 + (3+5) = (2+3) + 5$. Parentheses must be introduced in expressions involving several operations, for example $(8-4) + 2 = 4 + 2 = 6$ while $8 - (4+2) = 8 - 6 = 2$. Parentheses serve as a sign to "do this first". We employ certain conventions to avoid a cumbersome use of parentheses. For example, additions and subtractions are performed in the order in which they occur, as are multiplications and divisions. Multiplications and divisions are performed before additions and subtractions unless otherwise specified by symbols of enclosure. For example:

> $8 - 4 + 2$ always means $(8-4) + 2$. To express $8 - (4+2)$ the indicated parentheses must be used. On the other hand $8 + 5 - 2$ means $(8+5) - 2 = 13 - 2 = 11$. Here the alternative $8 + (5-2)$ is also 11.

Another example is

> $6 + 4 \times 3$ which means $6 + (4 \times 3) = 6 + 12 = 18$.

To indicate $(6+4)$ times 3 we must write $(6+4) \times 3 = 10 \times 3 = 30$. Consistent use of parentheses to indicate order of the operations establishes background for later formulation of rules. Usually the need for brackets and braces to indicate that the terms enclosed are to be treated as single terms does not occur until the later grades.

Relationships between mathematical functions appear in sentence form. A particularly important example is provided by the mathematical sentences used to characterize the function which expresses the distance between two points. This example is suitable for advanced students in Grade 7 - 8.

Let two points be a and b. Let the distance between the two points be denoted by $d(a,b)$. For example, if a and b are opposite vertices of a square of side 1, then $d(a,b) = \sqrt{2}$. The function $d(a,b)$ is characterized by three properties expressed by the following three mathematical sentences:

(i) $d(a,b) = 0$ if and only if $a = b$
(ii) For every two points, a and b, $d(a,b) = d(b,a)$
(iii) For every three points a, b, and c, $d(a,b) \leq d(a,c) + d(c,b)$.

Property (iii) states that for any three points a, b, c (for example see Figure 7) the distance between a and b is less than the sum of the distances between a and c and b and c.

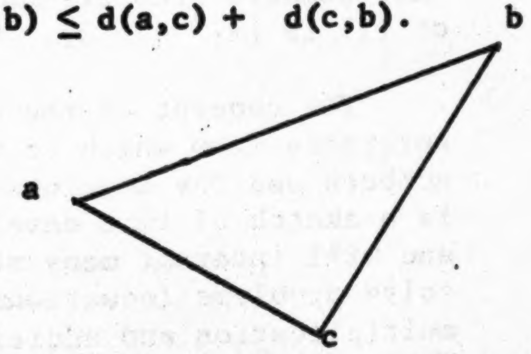


Figure 7

When a, b, c are points on a number line (see Figure 8) then $d(a,b) = |a - b|$. Then (iii) becomes the important relation

$$|a - b| \leq |a - c| + |c - b|.$$

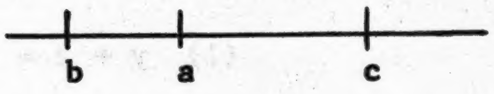


Figure 8

In the upper grades, 6, 7, 8, it will be important to analyze and solve more complicated mathematical sentences in a single variable. This may be done from the point of view of solution sets, or as a series of equivalent mathematical sentences. Here is an example:

For what numbers x is $2x + 3 < (2 + 5x) - 8$?

(1) $2x + 3 < (2 + 5x) - 8$.

Now consider the right hand side of (1). For all x, $(2 + 5x) - 8 = 5x + 2 - 8 = 5x - 6$. Hence for every x, (1) is equivalent to

(2) $2x + 3 < 5x - 6$.

That is, for every x, (1) is true if and only if (2) is true. We can also say that (1) and (2) have the same solution set.

(3) $2x + 3 + 6 < 5x - 6 + 6$.

(2) and (3) are equivalent and (3) is equivalent to

(4) $2x + 9 < 5x$.

Next (4) is equivalent to

(5) $-2x + (2x + 9) < -2x + 5x$.

Now for all x, $-2x + (2x +9) = (-2x + 2x) + 9 = 9$. For all x, $-2x + 5x = (-2 + 5)x = 3x$. Thus (5) is equivalent to

(6) $9 < 3x$.

Finally (6) is equivalent to

(7) $3 < x$.

Thus (1) is equivalent to (7) and the answer to the original question is: "All numbers greater than 3" or $\{x: x > 3\}$; equivalently the solution set of (1) is $\{x: x > 3\}$.

The concept of the mathematical sentence provides an important frame of reference from which to view the development of the integers from the whole numbers and the development of the rational numbers from the integers. Here is a sketch of this development which should be of interest to all teachers and will interest many students in grades 7 and 8. Basically, we need to solve problems (equations) which involve division and subtraction as well as multiplication and addition. Often these problems are of such a complexity that neither the system of whole numbers nor the integers is adequate.

We can of course solve equations like

(1) $y + 2 = 6$ and (2) $2x = 6$

within the system of whole numbers.

We need however to solve problems such as

$$(3)\quad y + 2 = 1 \qquad \text{and} \qquad (4)\quad 2x = 1.$$

The system of integers provides a solution for all equations of the form

$$y + c = d$$

where c and d are whole numbers. The integer which is the solution is denoted by $d - c$. The defining property of this number is that $(d - c) + c = d$.

The system of rational numbers provides a solution for all equations of the form

$$ax = b \quad (a \neq 0)$$

where a and b are integers. The rational number which is the solution is denoted by $\frac{b}{a}$. The defining property of this number is that $(\frac{b}{a})\ a = b$. The symbol $\frac{b}{a}$ is called a fraction; it is a numeral, a name for a rational number.

The problems which now arise in the development of the rational numbers are these:

- When does $\frac{b}{a} = \frac{u}{v}$?
- How are rational numbers added and subtracted?
- How are rational numbers multiplied and divided?
- How can a greater-than, less-than relation be introduced within the set of rational numbers?

Answers to these questions can all be motivated from the point of view that a rational number is a solution to an equation. For example: How should rational numbers be added?

Given $\frac{a}{b}$ and $\frac{u}{v}$, what is $\frac{a}{b} + \frac{u}{v}$?

We know that $\frac{a}{b}$ is a number x for which $bx = a$.

We know that $\frac{u}{v}$ is a number y for which $vy = u$.

We are asking for $x + y$. We need to find an equation for which $x + y$ is a solution.

Now $bx = a$ is equivalent to $v(bx) = va$ and
$vy = u$ is equivalent to $b(vy) = bu$. Hence adding these two equations yields

$$vbx + bvy = va + bu \text{ or}$$
$$bv(x + y) = va + bu.$$

Thus $x + y$ is a solution for the equation $(bv)z = (va + bu)$ and so it follows that $x + y = \frac{va + bu}{bv}$, or that $\frac{a}{b} + \frac{u}{v} = (\frac{va + bu}{bv})$

It is an important theorem that having all rational numbers, positive and negative, we can now solve all equations of the form

$$rx = s \quad (r \neq 0) \quad \text{and} \quad u + y = v$$

where $r, s; u, v$ are any rational numbers.

The development of the rational numbers must demonstrate these structural properties. It is equally important that rational numbers and the four fundamental operations be viewed in a physical context. Thus to answer "when we measure $2\frac{1}{2}$ yards in feet, how many feet do we get", we divide the number of yards by the number of yards in a foot. That is

$$2\tfrac{1}{2} \div \tfrac{1}{3} = (2 + \tfrac{1}{2}) \div \tfrac{1}{3} = (2 \div \tfrac{1}{3}) + (\tfrac{1}{2} \div \tfrac{1}{3}) = 6 + \tfrac{3}{2} = 6 + 1 + \tfrac{1}{2} = 7\tfrac{1}{2},$$

or

$$2\tfrac{1}{2} \div \tfrac{1}{3} = \tfrac{5}{2} \div \tfrac{1}{3} = \tfrac{15}{2}.$$

The search for solutions to equations will be continued throughout the entire mathematical program. Equations like

$$x^2 = 2$$

will bring irrational numbers. Equations like

$$x^2 = -1$$

will bring complex numbers. Functional equations like

$$f(xy) = f(x) + f(y)$$

will bring logarithms. Indeed, placing a spotlight on equations provides a guiding light to all of mathematics.

Strand 2. Geometry

The world around us is highly dependent on geometric shape and form. The interrelations of these geometric forms affect and structure our thoughts and actions. The universality of this geometric presence means that geometry must be an important part of the mathematics program for Kindergarten through Grade Eight. Everyone must know, for example, the significance of congruity and similarity to mathematics and its applications. Early experiences should develop an intuitive grasp of geometric concepts which will contribute to the success of later studies in geometry. A singularly important role is played by coordinate geometry as it fuses arithmetic, algebra, and geometry into a primary tool of science and mathematics.

Because geometry is universal every child can find many physical models of geometric concepts. Because these models are readily available and because they have a demonstrable utility, the learner will find geometry easy and interesting. A study of geometry will help each child to appreciate the role of geometry, its patterns, shapes and forms, in everything around him. The language of geometry enables him to communicate what he perceives to others. The ability to perceive a given geometric situation will develop his powers of analysis and classification and the utilization of known information in new situations. These abilities contribute to the ever increasing eclectic body of skills needed for problem solving. Therefore, geometry in all of its phases is an essential part of the total mathematics program in the elementary school.

The geometry program in the elementary school is a program of informal geometry. Here the word "informal" refers not to a casual manner of presentation or emphasis but to the absence of a formally developed subject using an axiomatic approach and stressing formal proofs. It should be the goal of the geometry program in grades K-8 to provide the foundation for this type of study. When appropriate, it will often be useful to present short deductive arguments.

An important part of the instructional program is the use of physical materials which have been carefully selected to enable pupils to have experiences with specific geometric concepts. A perspective drawing of a rectangular prism in a geometry text will have more meaning to the learner if he has first held a block or a box in his hand, described its various parts, analyzed it to determine what parts he would have to make to construct a similar box, and then actually made the model. Reconstruction activities lend themselves to the utilization of problem solving skills.

Here are a couple of examples which require this sort of analysis:

1. Draw a polygon with exactly seven sides. (For the purposes of a primary pupil, a polygon is a (closed connected) figure with straight sides drawn on paper.) It is interesting to contrast this problem with instructions to designate seven points A, B, C, D, E, F, G on the paper and then to connect "adjacent" points, A - B, B - C, ..., F - G, G - A, with

straight lines. A bonus to this problem is the discussion of convex and non-convex polygons.

2. In how many different ways may six squares with equal sides be joined, side to side, so that the resulting six squares may be folded up into a cube? For example can the six squares of Figure 1 be folded into a cube?

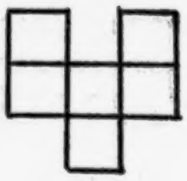
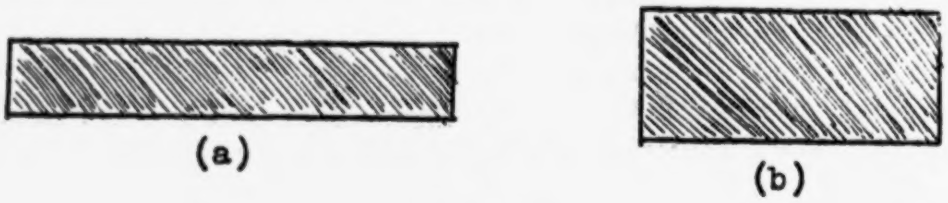


Figure 1

The language of geometry is an important part of classroom instruction. In the early years, the introduction of geometric terms should be quite informal. For example, in the early grades, as the pupil meets geometry for the first time, teachers should accept the term "triangle" for "triangular region". However, if teachers use the correct terms, then pupils will eventually become precise in using descriptive geometric terms. Pupils should also learn to describe a geometric shape or form in relative or approximate terms.



(a) (b)

Figure 2

For example, in describing the two rectangular regions in Figure 2, the pupil should be able to first recognize that both figures are bounded by quadrilaterals which are rectangles. Second, the pupil should be able to describe (a) as having one side which is very much longer than the other and (b) as having one side just a little longer than another. Following this initial intuitive approach, the use of formal descriptive terms utilizing line segments will prepare pupils for a more sophisticated abstract analysis.

The properties of geometric figures can be presented through experiences in classification. Classifying geometric shapes according to size and shape makes pupils aware of similarities and differences between these shapes. Further classification, based on other properties, will permit pupils to analyze a geometric shape according to the properties of a triangle, square, rectangle, parallelogram, pentagon or hexagon. These properties are often used in measurement. For example, in finding the perimeter of a regular polygon such as a square only one side needs to be measured if one is aware that all sides are the same length.

Other properties find interesting applications in, for example, construction: Why does a carpenter, in checking a rectangular frame, measure the length of the two diagonals?

Pupils should have experience interpreting pictures of geometric shapes and forms in both two and three dimensional space. Construction of geometric shapes with a straight edge and a compass should be included in the instructional program. The making of a pictorial model should be encouraged even though there may be an adequate model in the textbook. Such construction often provides clues to the solution of problems. As a part of the study of polyhedra we suggest an informal treatment of Euler's formula. This can create a new and exciting facet to the study of classifying geometric configurations. This formula relates the number of faces, F,; the number of edges, E; and the number of vertices, V, of a polyhedron: $F + V = E + 2$.

Once pupils have had many experiences with the geometry of the physical world in which they have been able to touch and see actual models of the concepts being developed, they then have the sophistication to enter into a study of geometry that cannot be directly visualized. In some respects, one might call it "make believe" geometry for the pupils have to perceive such concepts as a point, line, and plane as ideals that exist but cannot actually be represented in totality by a physical model. Primary pupils usually have difficulty bringing true meaning to the concept of a point since it is something they cannot see or touch. They are often unable to perceive the set of all points in space or on a number line since their concepts of quantity may not extend to such infinite dimensions. This, in itself, may be justification for the delay of these abstract concepts until grades 3-8, although in a spirally developed curriculum, learners in grades 1-3 should have initial experiences that intuitively develop the idea of a point as a fixed location and a line as a set of points which is infinite. The study of maps in the early grades provides many opportunities for this intuitive development as does the use of the number line.

The study of such abstractions as point, line, ray, line segment, plane, and angle, shows they are related to models in the physical world. Every attempt should be made to clarify these concepts through physical and pictorial models. Although it is difficult to make blanket statements for all pupils as to what they should know at a given level, it is assumed that, for example, pupils by the end of Grade Six will be familiar with and be able to describe a given line segment as the subset of a line ($\overleftrightarrow{AB}$) which is formed by two endpoints (A, B) and all of the points between them.

A basic concept in geometry is that of congruence. The concept of congruence grows out of classification activities and of certain experiences in the measurement of angles, line segments, the area of geometric shapes, and the volume of geometric solids. Two geometric forms or figures are said to be congruent if one can be superimposed on the other by a rigid transformation. (A rigid transformation is one which preserves the distance between two points. Such a transformation can be accomplished by a sequence of rotations about a point, reflections about a line, and translations. In particular no stretching or squeezing is permitted.) Pupils should understand the difference between congruence ($\simeq$) and equality (=) as it applies to geometry since this distinction is often required in the axiomatic approach in later geometry.

Pupils should have experiences in comparing similar geometric figures. They should see the relationship of the radii in each of several increasingly larger circles. They should be involved in problems which require them to see the relationship between the size of the radius and the circumference or area of the circle. In studying the volume of cylindrical containers, pupils should see the effect of changing the height as compared with changing the radius of the base. They will begin to see that if the height doubles, so will the volume whereas if the radius is doubled the volume is quadrupled. Such experiences introduce the concept of ratio and proportion. These are enhanced by construction of similar figures (Draw a traingle that has the same sized angles, but with sides twice as long as those of the given triangle.) Drawing maps to scale also illustrates the idea of similarity. More generally scale drawings lead to an understanding of similarity and yield interesting applications of mathematics to physical situations.

Many elementary classroom experiences in science and social sciences provide uses for the coordinate plane. Graphing of data provides pupils with informal experiences in working with ordered pairs and graphing these ordered pairs on a coordinate plane. Mathematical activities with the coordinate plane provide pupils with opportunities to solve problems, translate mathematical information from a table to a graph (or vice versa), and to practice basic facts of arithmetic in game-like situations.

We recommend that coordinate geometry be introduced at the primary level. Primary pupils can plot points in the first quadrant and graph data recorded in science experiments. Successive experiences will involve all four quadrants and by grade eight pupils should be able to graph linear and quadratic equations as well as write the mathematical sentence that describes the graph. These simple concepts of Cartesian geometry have many applications in real life and provide the best possible basis for pupil choice in the study of more sophisticated mathematics at a later time.

Metric geometry brings the strands of geometry, operations, and measurement together. Determining length, perimeter, area, volume, and angular measurement provides learners with a basis for the application of concepts and skills presented in the other strands. Not only should pupils have an opportunity to measure geometric shapes and figures with standard instruments such as the ruler and compass, but they should also have experiences in which they have to make general measurements. In the latter, the measurements will be expressed in terms of greater than or less than relationships without a quantitative designation. General measurements are made by visual analysis such as placing the model of one angle on top of another to determine whether one angle is larger or smaller than the other one.

Strand 3. Measurement

Measurement is an integral part of everyday life, so much so that we are often unaware of its extensive use and involvement in our thoughts, observations, and decisions. It is a key process in the applications of mathematics since it is a connecting link between mathematics and our physical and social environment. For these reasons it is vitally important that the elementary school program include a detailed study of the measuring process. As a pupil learns about measurements, he should be actively engaged in activities that utilize both standard and arbitrary units of measure. Since the process involves numbers, especially an extensive use of rational numbers, a large part of its study should be included in the elementary school mathematics curriculum. However, a substantial attention should be given to measurement in other subjects, such as geography and science.

Measurement is a process whereby numbers are assigned to objects or a set to represent certain quantitative facets of the object or set. For example, the following questions require measurement for an answer: How wide is this table? How big is a page of this book? How heavy is this rock? How many words are on this page? How many seats are in this classroom if there are five rows with six seats in each row?

The introductory stage of the study of measurement consists of becoming familiar first with the sets to be measured, such as line segments, solids, weight, time periods, and then with such common units of measurement as the inch, foot, yard, meter, mile, pint, quart, gallon, liter, ounce, gram, pound, gigafirken, minute, hour, day, week, month, and year. Familiarity with these things and units of measure is gained through a variety of experiences in seeing and feeling. Initially, comparisons are at a "greater than", "less than", or "equal to" level. The refinement and complexity of the various units of measurement determine the grade in which each one is introduced in the school program.

The analytical stage, which follows the recognition and identification stage, can begin with the development of some fundamental concepts. It is important to recognize that certain types of measurement may lend themselves more readily than others for a particular purpose, and that the decision as to which type of measurement is most applicable is based upon the pertinent physical attribute of the object. For example, length is an essential property associated with a line segment. Thus measurement associates a number with a line segment; this number is called the length of the segment. The measure of a segment, a number, tells how many times a "unit" segment can be fitted into the segment being measured.

In the initial stages, the "unit" chosen is some handy object, for example, a pencil, an eraser, a piece of string, the span of a hand, or a bucket. However, pupils should soon be led to see that while the choice of a "unit" is arbitrary and may be a locally accepted way of measuring, it is necessary for purposes of accurate communication to choose one or more standard units.

An understanding of the approximate nature of measurement is essential. Exact measurements exist as ideals and are obtainable only in discrete situations in which the measurement process consists of counting the number of elements in the set to be measured. When a segment is measured, a scale based on the unit appropriate to the purpose of measurement is selected. Every measurement is made to the nearest unit. By using smaller units, more precise measurement is obtained. Inherent in a measurement process are two types of errors. One is caused by the instrument employed in the measurement. For example, have a pupil measure the length of a table top once with a foot long ruler and once with a yard stick; (or with a ruler scaled in centimeters and a meter stick.) Expect different results! Another error is introduced by the person making the measurement. As an instructive exercise ask each member of the class to measure the width of the classroom to the nearest quarter inch using a foot-long ruler. It is most likely that a large number of different answers will arise and a decision as to the best approximation to the length will involve statistical procedures.

Approximations in measurements should be related to estimations of numbers in arithmetic, so both of these underlying notions reinforce each other. Relevant to estimation, insight into the operations should involve much more than the obtaining of an answer. Pupils should have experiences in which they have to make an estimate of a product or sum based upon their knowledge of the nature and order of numbers. For example, in the problem, 5×193, a pupil should be able to estimate that the product will be less than 1000. He is able to substantiate this by thinking that 193 is almost 200 and $5 \times 200 = 1000$.

A sense for number should be nurtured so that the learner does not make blind estimations. For example, to estimate $\frac{69}{23}$ the conventional rule for rounding off replaces 69 by 70 and 23 by 20, so that $\frac{69}{23}$ is estimated as $\frac{70}{20} = \frac{7}{2}$. Whereas estimating $\frac{69}{23}$ by $\frac{60}{20} = \frac{6}{2} = 3$ yields a better estimate. In fact, in this case it yields the correct answer. In the first estimation, increasing the dividend has the effect of increasing the quotient and decreasing the divisor further increases the quotient. On the other hand, in the second estimation the effects of the estimation are compensatory.

Associated with these principles in numerical calculations are the problems of escalating errors in measurement through the use of mensuration formulas. For example, suppose that the sides of a rectangle have been measured and found to be 124 centimeters and 32 centimeters; measured to the nearest centimeter. What is the area? What errors are introduced by using the formula $A = L \times W$? We know only that $123.5 \leq L \leq 124.5$ and that $31.5 \leq W \leq 32.5$. (See Figure 1)

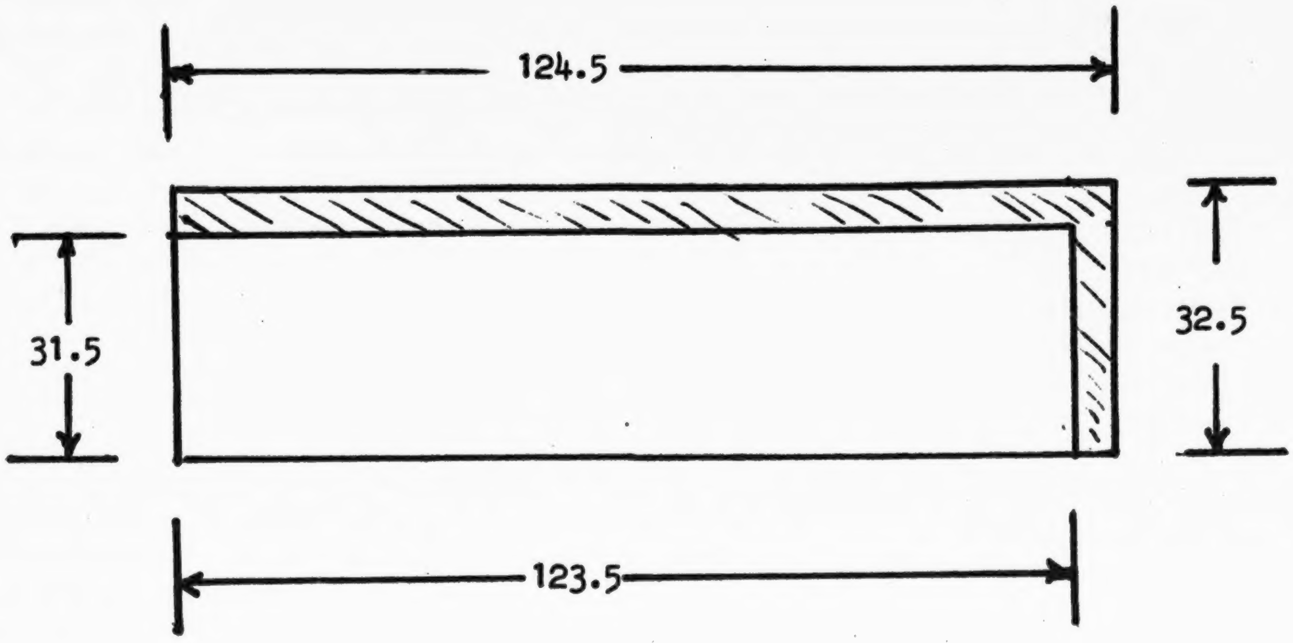


Figure 1

Thus $123.5 \times 31.5 \leq A \leq 124.5 \times 32.5$. Here the difference in the extremes is 156 so that use of this formula may give a measurement which may be in error by as much as 156 square centimeters. The product 124×32 might be estimated as $100 \times 30 = 3000$. A better estimate is achieved by compensatory approximations: $100 \times 40 = 4000$.

In addition to the common units of measurement the pupil encounters in his immediate environment, he should become familiar with the metric system. Metric systems of measurement are commonly used in the natural sciences and are being generally accepted throughout the world. One of its advantages is the ease of converting from one metric unit to another metric unit. Moreover, the relation to the decimal system of numeration identifies the metric system as a "natural" system to use. For example, a length measuring 3 meters, 8 decimeters, 4 centimeters, is expressed in centimeters by the place value numeral 384. After an introduction to the metric system, both the metric and English systems should be used interchangeably and frequently so that the metric system will become as familiar to children as the English system. In this respect, the children can become bilingual. We do not recommend a major emphasis be placed on the conversion between the metric system and the English system. Rather, approximate relationships between frequently used units might be established; for example, the concept that a meter is a little longer than a yard, or that an inch is a little more than two and one-half times as long as a centimeter. We recommend especially that every pupil have the opportunity to make actual measurements with actual meter sticks, gram weights, and liter containers as well as yard sticks, ounce weights, and pint containers.

For convenience in computation, and to express the degree of precision claimed for a measurement and ease in comparing two measurements, scientific notation is important. It is to be expected then that scientific notation, including negative exponents, will be studied before the end of the eighth grade.

The same conceptual sequence used in the measurement of line segments can be followed in the measurement of angles, areas, and volumes. For simple angles, this starts with developing an intuitive device for comparing the size of a pair of angles. Thus an angle is "larger than", "smaller than", or "the same size as" another angle. Models of the angles may be made of wire so that they can be superimposed, or one angle may be copied onto another. The basic test for comparing two angles emerging from this approach is: "When two angles have a vertex and side in common and the other side of one angle lies in the interior of the other, the angle whose interior contains the side of the other angle is the larger." Angles may be measured with arbitrary unit angles by "filling in" their interiors with these unit angles. Again the measure is to the nearest unit. Markings on a clock suggest that circles can be divided into subsets of equal measure and thus used to measure angles. This leads to the development of angles of standard unit measure and thus to the construction and use of a protractor.

For the measurement of area and volume, a start is made by identifying the interior regions of plane figures and solids and developing intuitive means of comparing the sizes of these interiors. In the case of area of plane figures, models can be made of the interiors to be compared and the pupil can see if one can be fitted inside the other, cutting one of the models into pieces if necessary. Similar procedures may be used in the case of solids. Solids may be thought of as containers, and, using the same intuitive approach, the volumes of the solids may be compared by finding which model holds more water or sand. Areas or volumes may be measured with arbitrary units by finding how many of such units are needed to completely fill the interiors of the plane figures or solids. In any event, whether one or the other measurement--such as area or volume--is to be used, again depends upon the physical property considered most suitable for the purpose. Graph paper may be used to advantage in finding the areas of plane figures. Pupils should gain an understanding of why standard units in the form of squares or cubes are chosen for measuring area or volume, respectively. The development and use of formulas for the calculation of areas and volumes is particularly important at this time.

Measurement of other phenomena such as weight, time, and heat can be studied in a similar fashion. Computation with measures and conversion of units should evolve from actual situations (as distinct from verbal descriptions of actual siutations) in order that it be done meaningfully and the results interpreted reasonably in terms of significant digits. Ample opportunity should be provided for estimation so that the teacher can determine if the process has meaning for the pupil.

The development in kindergarten and grades one through eight of the Measurement Strand should lead to general concepts of measurement such as the following:

- Measurement is a comparison of the object being measured with a "unit" and yields a number to be attached to the object as the measure of the object. (Measurement may thus be conceived as resulting in a pairing of objects with numbers. This pairing is of a type that leads to the notion of a function. Therefore measurement may be treated as a special case of a function.)
- The choice of a measurement "unit" is arbitrary, but standard units are agreed upon for accurate communication and simplified computation.
- Measurement is approximate, and the precision of the measurement depends upon the measurement unit employed.
- Any process of measurement has the following basic properties:

  If object A is part of object B the measure of A is less than or equal to the measure of B.

  If objects A and B are congruent, then their measures are equal.

  If objects A and B do not overlap, then the measure of the object consisting of the union of A and B is the sum of the measures of A and B.

It is important to identify these basic properties as they are used in developing the mensuration formulas. For example, the development of the formula for the area of triangle (as expressed in standard units)

$$A = \frac{1}{2} b \cdot h$$

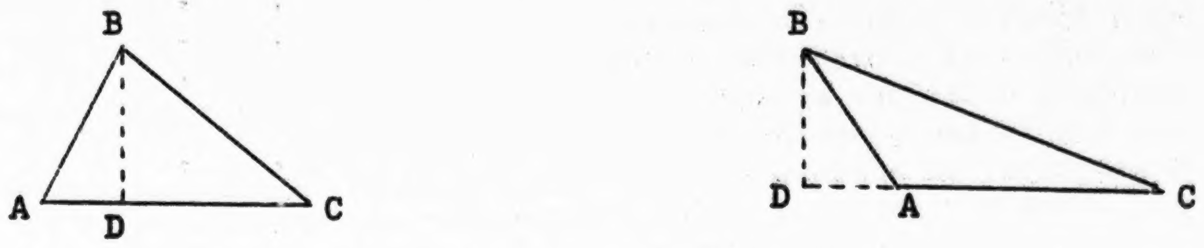
can be indicated by a cutting and matching exercise together with the properties mentioned above.

First we consider a right triangle as shown in Figure 2.

Figure 2

Construct a rectangle of width h units and length b units. Cut it along a diagonal. By placing these two triangles together on T we show that all three triangles are congruent. Thus each has the same area. Moreover, since two of them exactly cover a rectangle of area $b \cdot h$, it follows that each triangle has an area which is expressed by $\frac{1}{2} b \cdot h$.

Now in a standard fashion as indicated in Figure 3 we obtain the same formula for the area of any triangle by computing the sum (or difference) of two right triangles.



(a) (b)

Figure 3

Let ▲ ABC denote the triangular region determined by the triangle ABC. Then corresponding to Figure (2a)

▲ ABC = ▲ ABD U ▲ BCD

and corresponding to Figure (2b)

▲ ABC = ▲ BDC - ▲ ABD

Another interesting problem is the construction of a ruler. This problem also serves as an example of a problem whose solution does not depend on a numerical calculation.

The problem asks each pupil to construct his own ruler using the following materials furnished by his teacher: A strip of heavy material such as a blank wooden straightedge, a strip of file folder, etc., and a sheet of ruled binder paper. (The blanks should be of different lengths, and the ruled paper should not be the same length as the blanks.)

Pupils should be given free rein in their approach to this problem, and the idea of unit lengths should be discussed thoroughly before they start. Care should be exercised in the assignment so that a solution will not be immediately available to some, and so that the frustration of the less mature pupils will not become overwhelming. For example: The statement of this problem probably is not sufficiently specific so that we could anticipate almost immediately the question, "How many parts?". If the number assigned is either a multiple of two or three, many pupils will arrive very soon at a solution by simple folding of the binder paper. The number twelve will seem a reasonable answer to the above question, but a tryout on adults will show that this is too simple. (We might ask ourselves why this number might be a reply?)

In the preliminary discussion we should help pupils to see that units are arbitrary, and that each may select his own unit and name it for himself. If he does this, will he arrive at a "correct" answer to a future problem in measurement? Here we can stress the fact that the customary units were chosen and prescribed to facilitate communication.

The solution given below in Figure 4 will probably be the most satisfactory, and will generally be found fairly soon by at least one pupil.

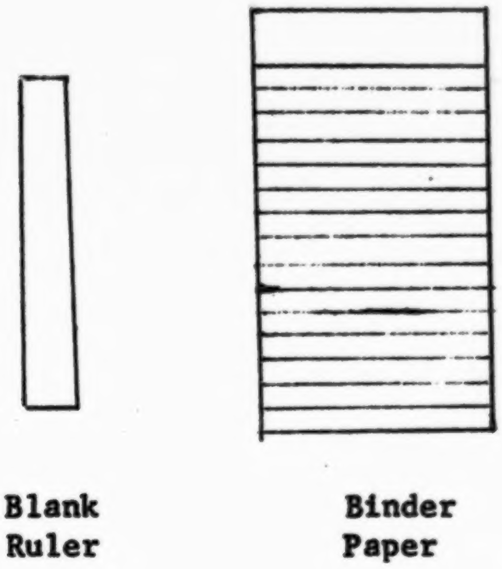


Figure 4

Solution: To divide the blank into ten equal parts, count ten spaces on the binder paper and lay the blank diagonally on the ruled lines so the ends match ten spaces. Each line will then mark one division.

As a final exercise, give a measurement assignment, for length only, and have pupils compare results. Who is correct? Have pupils exchange rulers and compare results. Such an exercise can lead to new insights into the measuring process. This approach probably will give the pupils a more sophisticated understanding of measurement than that of many adults. Further refinements can be carried out to the extent of their usefulness at a particular grade level.

Strand 4. Applications of Mathematics

Children have a natural curiosity about phenomena found in the physical world. As they work with physical materials they unconsciously pose problems that need solution or ask questions such as "Why does this work?" Such an approach to the teaching of mathematical concepts is not only more interesting to pupils, but provides them with practice in basic skills of arithmetic and geometry. Practice of this type has a special meaning to each pupil since he is able to relate it to something that he has been doing.

There is an important point to be made in the application of mathematics. Mathematics does not deal directly with the raw physical situation. For example, mathematics does not divide 10 apples by 5 children. Mathematics deals only with a refined mathematical model of the situation. Mathematics divides the number 10 by the number 5. We interpret the answer, 2, as meaning, for example, that if 10 apples are evenly distributed to 5 children then each child will have 2 apples. While it is technically incorrect to divide 10 apples by 5 children and obtain 2 apples per child as an answer, this procedure, is one which follows standard scientific practice in the determination of units such as "miles per hour". We recommend that the distinction be made a part of the teacher's manual so that he will appreciate the process. In more complex situations, especially when the model does not fit the situation so exactly, the distinction between the model and its origin will be crucial.

In this strand, Applications of Mathematics, we are concerned with problems which arise in the context of some natural event. Often this will be a phenomenon arising in the natural sciences, (astronomy, biology, chemistry, engineering, geology, meteorology, psychology, physics, etc.), in the social sciences, or in the humanities. We have in mind problems which will be so meaningful to each pupil that he may have an honest expectation of experiencing the problem himself. To the child who looks at a tree and says "That's a tall tree", the question "How tall is that tree?" will have meaning. Specifically, we exclude from consideration in this Strand all those problems which build proficiencies in mathematical skills as well as the contrived (albeit, interesting) word problems. Treatment of these classes of problems we include in Strand 9, Problem Solving.

The rationale of applied mathematics runs like this: (1) Given a concrete situation for analysis, a study identifies the features which are significant to the central problem. This analysis may be informal or transparent; on the other hand, it may be quite complex. (2) Those features which are amenable to mathematical analysis are abstracted and relations between these mathematical quantities are expressed in a way which characterizes those in the original concrete situation. Any other features are ignored! (3) The abstract formulation of the problem is then subjected to a mathematical analysis. Hopefully one can be given! If not, then additional simplifying assumptions and modifications of the model must be made. Strategies for mathematical analysis will be discussed in Strand 9, Problem Solving. (4) Finally the mathematical solution is reinterpreted in the light of the original problem. Predictions for future behavior of the situation are made.

The abstract formulation is called the mathematical model of the problem. The validity and usefulness of the model is judged by how well it can predict future behavior in actual practice.

We recommend that the steps outlined above be discussed in the teacher's manual of the textbook series. However, they should never be presented as a problem solving plan which must be articulated and detailed in each problem. In particular, each pupil and teacher should feel free, and should be required often, to rephrase the initial problem, reselect the properties to be modeled, reconstruct the model, and redo the analysis. We should never permit formalistic procedures to block intuition or flashes of genius. Intuition should be strengthened at every opportunity by encouraging informed guessing and estimation. On the other hand, we caution that pupils are often confused and inhibited by a failure to make explicit some of the steps we have outlined here. Those who can easily do the mathematical analysis may be totally unprepared to construct the correct mathematical model.

So far we have stressed the strategy of model building as leading from physical problems to more abstract mathematical models and their analyses. The reverse process is pedagogically important: Given a mathematical problem, construct a physical realization of the problem. Such a physical model of a mathematical problem is a significant aid to understanding mathematics. Indeed, in the first growth stage of mathematical application a mathematical concept is developed through varied experiences with specific objects and situations that are interesting to pupils. As the pupil uses a concept in a variety of different concrete situations, its abstract nature gradually evolves. Growth in ability to make generalizations is very important and should not be neglected. Generalizations should not be formalized prematurely.

The nature of the application of mathematics will vary greatly from kindergarten through grade eight. We recommend that at each level applications include not only those in which the model is arithmetical but also geometrical. We recommend that applications be selected which use mathematical principles already developed and others which motivate new principles. As often as possible applications should be based on physical situations (which the learner may experience himself) rather than on conversation about hypothetical siutations.

As the age and maturity level of the pupils increase, the sophistication of the problems approached will be limited only by the opportunities pupils have to participate in situations which have real problems that need solution. The approach to a specific problem should evolve from a self-determined pattern, as derived by the pupils from their previous mathematical experiences.

Here are four examples of the modeling process; each is a child-sized version of a practical problem.

> Example 1. School is four blocks from Paul's home; one block north and three blocks east. How many different routes may Paul choose to walk to school? We first make a map of the local area. (This in itself is a modeling process. The map in Figure 1 does not consider the fact that a sidewalk exists on both sides of

each street. Thus it ignores the complication of distinguishing between two routes which use the same streets, but use different sidewalks!) We introduce

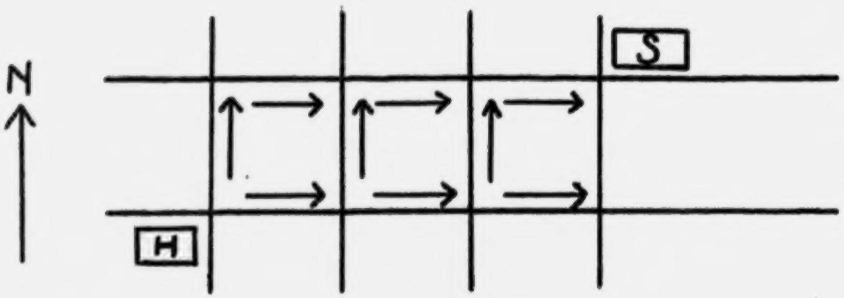


Figure 1

the simplifying assumption that Paul will not "go out of his way" nor will he take a shortcut. Mathematically we can interpret this to mean that the length of his route shall always be four blocks long and always follow streets. Now the analysis may be done from the map. The figure above uses arrows to show the alternatives at each corner. The analysis can be completed by counting the possibilities.

We should point out that this counting argument which consists of first systematizing and then counting all possibilities is an extremely useful type of application. For the K-8 pupil, it will provide important readiness for later work in probability.

Example 2. Carpeting is to be laid to cover the classroom floor. How much carpeting should be ordered? We shall need to know something about the shape and area of the room. We also have to know something of how carpeting can be purchased, i.e., by the square foot, square yard, or by the linear foot from a roll. We may also need to know how the pattern, if any, will affect the way in which the material can be cut. These considerations we discard. We work with a simplified problem of determining the area of the room. To determine the area we again make a further simplification: the floor is rectangular in shape. This may ignore special features of the room such as an entry way or a closet. Of course the room will not be a rectangle; it probably will not even have the exact measurements of a true rectangular shape. But we construct a geometrical model by representing the floor as a rectangular region. Upon actual measurement some other quadrilateral might appear as a better mathematical model, but we measure and assign lengths to the sides of the rectangle. Now we make the mathematical analysis (see the Measurement Strand) which computes the area. Finally this number must be interpreted as how much carpeting should be ordered. Whether or not the order is placed on the basis of this analysis depends on how exact the model is believed to be.

Example 3. An electric train runs around an oval track. What is its speed? Our first restriction is to discuss only the notion of speed as "average speed". Our assumptions require that the train has been running and so a "steady state" has been reached. We ignore changes in speed as the train rounds the curved ends. We assume that we can accurately determine the time of one circuit of the train around the track. We assume that the track is an oval. Thus we construct a geometric model of the track:

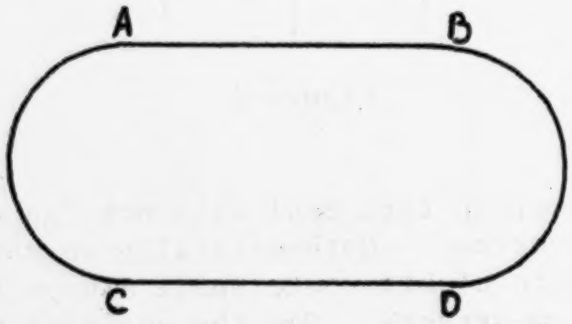


Figure 2

In Figure 2, line segments AB and CD are parallel, arcs CA and BD are semicircles. We make measurements. The first part of the analysis determines the circumference of the oval and interprets this number as the length of the track. Now with stopwatch in hand we time one complete run of the actual train around the actual track. (Or we might let the train run for several laps, time the total elapsed time of the run, and then compute an average time for one lap. We should then discuss how the experimenter reduces the measurement errors inherent in the assumption in order that we can determine the time of one circuit of the track.) Our analysis uses the fact that distance (d) traveled at a constant rate (r) of speed in (t) seconds is the product of $r \cdot t$ (the units representing r, multiplied by the number of units representing t). The distance traveled in time (t) can be expresssed as a mathematical function: $d(t) = r \cdot t$. It is important to point out that a graph of this function yields a straight line.

Example 4. How fast do bacteria grow? We have at our disposal some Petri dishes and some source material from growing various bacteria. (Here we envisage borrowing the science background from the science curriculum.) We attempt to determine the amount of growth at twenty-four hour intervals. (This is in itself a modeling exercise since the growth will have to be estimated from an estimate of the area of the region of the Petri dish covered by the growth.) These data are recorded on a graph as in Figure 3.

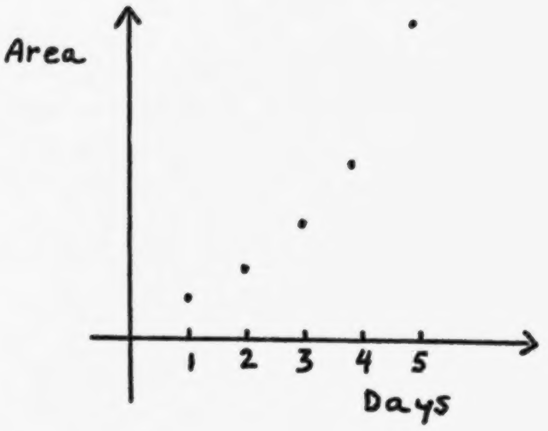


Figure 3

We ask: How can the growth of these bacteria be described? Is there a mathematical relationship which will help us to predict future growth? We ignore conditions such as the effect of imperfectly controlled heat and light, the amount of bacteria already present in the host material. We also assume that the growth is continuous and we draw a smooth curve between the plotted points (Figure 4).

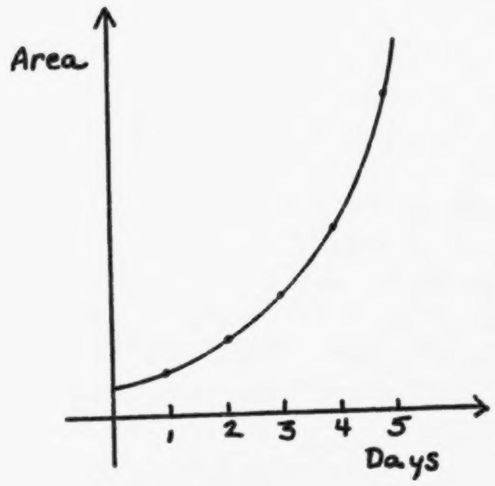


Figure 4

Now the mathematical analysis is of this graph; we seek a function which describes this growth. We might find that the amount of bacteria doubled each day and thus express the functional relation by

$$\text{Area after } N \text{ days} = k \cdot 2^N$$

where k is the amount initially present, or half the amount present after one day. In functional notation: $A(N) = k \cdot 2^N$. Further experimentation could be used to confirm the general correctness of this relation. On the other hand it might show that

$$A(N) = k \cdot 3^N$$

better describes the function.

The mathematical model for growth of bacteria then is a functional relationship which describes the amount of bacteria after a fixed time. If the model has validity we could predict, after determining the constant (k) by observing the growth during one period, what growth should be expected at any time during the growth cycle of the bacteria. It is important that from this experiment and analysis the pupil learns that growth is an exponential function of time, here denoted by the letter t. A functional relationship of the type

$$A(t) = k \cdot b^t$$

where b and k must be determined represents almost all growth processes. It is, of course, only a mathematical model since many of the factors controlling growth have been neglected. Basically it lies behind our population explosion. The "explosion" part exists because the function is of the type

$$A(t) = k \cdot b^t$$

and is not of the type

$$A(t) = b \cdot t + k.$$

Strand 5. Statistics and Probability

We live in a world of statistics. Advertising agencies bombard us with information indicating the superiority of their products and quote their findings. Governmental departments report findings to substantiate the need for expenditures. Medical research comes to us as a report of data that have been gathered in the furtherance of a better cure for a disease. Statistical data points to smoking as a possible factor causing lung cancer because there exists a high correlation between heavy cigarette smoking and the incidence of lung cancer. Newspapers and magazines contain graphs, tables, and charts and their stories reflect an interpretation of the data. The effective citizen of today and tomorrow needs to understand and interpret the statements and data with which he is confronted; he will have to make decisions based on analysis of such data. The permeating influence of statistics and its theoretical sister, probablity, compels students to understand enough about these aspects of mathematics in order that they intelligently assess information having undertones of either one or the other.

The ability to interpret data can come only when one has had many experiences in collecting data, learning how to organize the data so that conclusions can be made, and in interpreting data. Therefore, experiences in collecting, organizing, and interpreting data should be included in an elementary school mathematics program.

The collecting of data should be the outgrowth of experiences involving observations. The elementary classroom abounds with sources of data. The children, themselves, provide data regarding their names, their heights, their preferences, the books they read, their birthdates, or their summer vacation travels. Science experiences provide opportunities for two types of data--quantitative and observational. Quantitative data are those in which something can be measured or assigned a numerical quantity to denote change or growth. Observational data are of a descriptive nature and denote more of a general physical change; for example, the stages of development of a butterfly.

Data collecting should involve the pupil. As a pupil becomes involved in the collection of data, he becomes interested. He learns to be more observant as well as analytical. He becomes discriminating and examines discrepancies in the data he collects. His "mathematical eye" is often opened for the first time. He is also engaged in the problem solving process. Data collecting begins with the recognition of the sources of data once a purpose for collecting data has been established. Instruction should begin with data that have some direct relation to the pupil or that have been collected by the pupil. Often data found in textbooks have little meaning to the pupil. For example, it is of greater value to the pupil to collect the daily temperature readings for a period of time at school or home than it is for him to turn to a report in a book where similar data are available. The information he will be using is the result of something in which he, himself, has been involved. Once he has sufficient direct experience with collecting data, then data from textbooks can take on more meaning.

Organizing data is an art. It is something that one must learn. In order to read the information in a table, graph, or chart, it must be organized in such a way that it complements the purpose for which the data were originally gathered. Pupils need to learn how to organize data to tell a specific story. Part of the instruction in the elementary school mathematics program should be spent on learning the fundamental procedures for making tables, graphs, and charts as well as the many different types of each of these. Pupils should become familiar with the type of graph, chart, or table that can best be used with each of the different kinds of data that is collected.

Collecting and organizing data helps the pupil in understanding what a chart, table, or graph represents, but the greatest task comes when the pupil must read that graph, table, or chart and decide what story it really tells. This is central to what might be considered simple statistical inference.

There are many activities which may interest children in grades four, five, and six. For example, make a table of the heights and weights of the class members, or how far each can kick a ball. When the data are broken into boy-girl classifications some interesting conclusions appear. Compare the data (for height and weight) with the data for the whole school. A variation of this problem is to plot a point (height, weight) for each pupil in your class. What conclusions can be drawn?

Another excellent example is the average monthly temperatures in your city. Suppose the average monthly temperatures of a city, say San Francisco, were plotted in a graph with the months identified along the horizontal axis and the temperatures along the vertical. This graph has its lowest point around January, rises to a peak around August or September, then declines as it approaches December. Obviously, discussions can lead to the characteristic behavior for the winter and summer. Next, the same kinds of information may be presented with more than one city graphed on the same set of axes. A good choice of locations--say, San Francisco and San Diego--may reveal that while the general tendency (low in winter, high in summer, intermediate spring and fall) may be observed, a more striking observation may be made by comparing the data: one curve is more peaked and the other more shallow. The lesson may be extended by reviewing other data and observing that the range between maximum and minimum points is less for stations closer to the equator (local irregularities minimized), and statistics for some other stations in the equatorial regions may be produced to verify such a conjecture. To develop further the winter-summer theme, temperature graphs for stations in the Southern Hemisphere (say Sidney, Australia) may be studied.

Interpreting data involves a very critical and analytical study of the data. Pupils should be aware of such factors as sampling, the purpose of data collecting, the scale used in the construction of a graph, as well as such indications of central tendency as mean, median, mode, variance and standard deviation. Learning experiences should provide the pupils with opportunities for making predictions and for drawing conclusions from the data given. Pupils should learn to challenge the source of the data and the manner in which they are pictured.

We illustrate these concepts with four sets of data which presumably represents noontime temperatures on ten successive days in each of four California cities. (Preferably this data would have been read from a classroom thermometer.) For ease in studying the data we have listed the data in decreasing order. This is not important, but it helps serve in lieu of a graph and it will enable us to appreciate more easily what is happening.

	LA	SD	SF	SB
	100	80	72	75
	80	80	72	74
	80	80	71	73
	80	80	71	72
	60	80	71	71
	60	60	69	69
	60	60	69	68
	60	60	69	67
	60	60	68	66
	60	60	68	65
Totals	700	700	700	700
Ave. or Mean	70	70	70	70
Mode	60	60 or 80	69 or 71	Any number
Median	60	70	70	70

The median and the mode serve principally as quick approximations to the mean since little numerical calculation is required. In comparison with the mean they give us a little indication, as in the case of LA that more of the observations were below average than above.

These examples show that the average tells us little about the overall distribution. For example, many people might prefer the temperature distribution in San Francisco since it has very little variation and no extremes.

The next statistic, the variance, or its square root, the standard deviation, gives us an indication of the amount of spread in the data around the average. As we shall see, a great deal of spread, as shown for Los Angeles, will have a high variance while, only a small spread, as of San Francisco, will have a small variance.

To compute the variance, for each observation subtract the mean from the observation, square this number and add for all observations. Finally divide by the number of observations.

LA

Observation	Obs. - Mean	(Obs. - Mean)^2
100	30	900
80	10	100
80	10	100
80	10	100
60	- 10	100
60	- 10	100
60	- 10	100
60	- 10	100
60	- 10	100
60	- 10	100
Average 70	Total	1800
	Variance	180

Standard Deviation = $\sqrt{\text{Variance}}$ = 13.4

SF

Observation	Obs. - Mean	(Obs. - Mean)^2
72	2	4
72	2	4
71	1	1
71	1	1
71	1	1
69	- 1	1
69	- 1	1
69	- 1	1
68	- 2	4
68	- 2	4
Average 70	Total	22
	Variance	2.2

Standard Deviation = $\sqrt{\text{Variance}}$ = 1.48

Knowledge of the variance or the standard deviation indicates how much confidence we can place on the average. For example, in planning a trip to Los Angeles, or San Francisco should you take warm or light clothing? The average temperature for both cities is 70°. At San Francisco the variance is 2.2; with some confidence we can count on the temperature when we arrive to be close to 70°. But at Los Angeles the variance is 180; we cannot count on the temperature being close to 70°!

Experiences in collecting, organizing, and interpreting data can begin in the kindergarten and should be a part of the instructional program at each succeeding level through the eighth grade. Both simple data collection and organizing of data weave into the strand on functions; the first, by one-to-one correspondence between an element in a sequence and its statistic and the second, by the display of a table or of a graph. In the curriculum for grades 4-8, a more advanced step may be taken via interpretation of data. The introduction and use of formal statistical terms such as mean, frequency, median, mode, variance can begin informally for some pupils at the fifth or sixth grade level, and certainly should be studied by the end of the eighth grade.

Experimentation is a natural way to produce sets of statistics. For example, plotting the length of a suspended spring against the weight applied to the spring will show a straight line up to a certain limit (the limit is reached when the spring has been permanently deformed). Plotting growth of beans can give height as a function of the number of days from germination. Still another example may be used to illustrate the inverse of the squaring function: namely, obtaining a constant multiple of the square root via experiments with simple pendulums of various lengths:

$$T = \sqrt{\frac{\ell}{g}}$$

where T is the period, ℓ the length, and g a constant due to gravity.

Clearly, empirical data are not likely to yield a smooth curve precisely. Thus deviations or anomalies from uniformity hint at the need for curve-fitting procedures. The intuitive level of curve-fitting considered appropriate at this time needs not go beyond drawing the best curve that fits most of the plotted points.

The central role played by normal distributions in statistical problems should be explained at appropriate times. Many examples which have distributions of this type should be studied. Measurements of physical quantities constitute an important class of examples. These examples should display the important role played by the standard deviation in normally distributed populations since 95% of the population fall within the interval bounded by two standard deviations from either side of the mean.

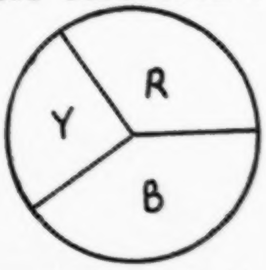
In statistics, a sample is produced and the principal task is to study the composition of the sample in order to arrive at an estimation of the composition of the original population from which the sample has been drawn. For example, a sample of 59 cars is taken from the assembly line producing 10,000 cars, and 3 of these 59 are found to have defective brakes. How many cars of the 10,000 produced in this assembly line are likely to have defective brakes?

Clearly, a model for this kind of problem is the following: 25 marbles are drawn from a large jar of 1500 red and white marbles. Of the 25 marbles drawn, 20 are red and 5 are white. What is likely to be the original composition of marbles in the jar?

Probability theory reverses this process in the sense that the probable composition of the sample is deduced from the known composition of the original population. For example, if it is known that 1% of all people in the United States are left-handed, how many people in a United States city with a population of 800,000 are left-handed? Or, it may be known that 80% of the crop in a field produces fertile seeds. From a packet of 200 seeds collected from this field, how many can be expected to germinate? Corresponding to these are models such as the following: In a container are 100 marbles, 80 of which are red and 20 of which are green. If a sample of 5 marbles is drawn from this container without looking (and at random), how many of these are expected to be red: If the original composition is 80 red and 20 green, then the probability for one marble drawn at random to be red would be $\frac{80}{80 + 20}$ or $\frac{4}{5}$.

The theoretical foundations for the reliability of statistical data for making predictions rests in the theory of probability. We do not plan to develop that theory in the elementary grades! However, some experience with the laws of chance is invaluable and necessary. One reason is that the mathematical models of many scientific and economic problems lie within probability theory. We recommend that the beginnings of this subject be developed in grades K-8.

Probability experiences in the elementary school can be provided by drawing marbles out of containers or by spinning spinners. Many children's games come supplied with spinners (spinning pointer to determine the number of moves a player might make, etc.). These can be adapted for classroom experiments in probability by varying the areas of the sectors. For example, in the one shown at the right, $\frac{1}{3}$ of the region is red (R), $\frac{1}{4}$ of it is yellow (Y), and the remainder is blue (B). What is the probability of the spinner pointing to blue on a spin?



Other experiments that can be performed in probability at the elementary school level include flipping of coins for heads-tails, and tossing of a die, varying the outcomes perhaps by coloring the faces of the die. For example, a die with 4 faces painted red, 1 face yellow, and 1 face blue, is likely to produce different results in tallying color of faces than one with 1 red, 2 yellow and 3 blue faces.

In the seventh and eight grades, these experiments can be increased in complexity. Some suggestions for such modifications leading to significant results are offered below:

(i) by considering series of repeated trials of the same experiment;

(ii) by combining outcomes of different experiments, for example, the result of a spinner together with the result of a tossed coin, or the result of Spinner A with the result of Spinner B, etc.;

(iii) in the case of drawing marbles from a container, by considering the effect of not replacing the drawn marble as against replacing the marble before the next draw.

Results of repeated tosses of a coin may be tallied. In a pair of tosses of a coin, the results may be

hh: heads on the 1st toss followed by heads on 2nd toss;
ht: heads on the 1st toss followed by tails on 2nd toss;
th: tails on the 1st toss followed by heads on 2nd toss;
tt: tails on the 1st toss followed by tails on 2nd toss.

Each pair of tosses may then be tallied under one of these four possible outcomes: hh, ht, th, tt. When the Hardy-Weinberg principles and the Mendelian law are discussed in the science class, results of the coin-tossing can be compared with the theoretical results in genetics.

Repeated tosses of a coin may be further extended by taking each triplet of tosses as a single outcome. The possible outcomes may then be given by: hhh, hht, hth, htt, thh, tht, tth, ttt, where, as before, thh would mean tails in the 1st toss, followed by heads in the 2nd toss, followed by heads in the 3rd toss.

All the ramifications in statistical inferences need not be formally treated until high school, but as in the case of coin-tossing experiments, an intuitive feeling for statistical inference may be acquired through a wealth of experiences in probability, leading to insight into the understanding, for example, of actuarial events.

With a carefully designed curriculum, it is possible and desirable to introduce aspects of probability theory to include dependent and independent trials, and aspects of statistics to include simple statistical inferences. Concepts of probability and statistics form an important foundation to the natural and physical sciences and to the social sciences and business as well.

Strand 6. Sets

The concept of a *set* is fundamental for communicating ideas in mathematics, just as it is in our everyday life. We speak of teams, committees, classes, groups, congregations, armies, flocks, herds, and so on. It is important that the concept of a set be introduced and used throughout the mathematics curriculum. It is particularly important that once the concept of set is introduced it be used effectively in subsequent mathematical development. Indeed, if no such use can be made, then the concept should not have been introduced. Like the playwright Chekhov, we believe that a cannon should not be brought on stage unless it is to be fired.

Throughout the K-8 curriculum we envision set concepts playing a major linguistic role. The language of sets should be used as needed to aid mathematical communication; to gain clarity, precision and conciseness. The symbols of set operations may be introduced to aid in this task by avoiding longer, more complicated, verbal equivalents. However, at no time do we suggest that set theory be developed as an end in itself. Thus we do not visualize including in the curriculum a treatment of Boolean Algebra or axiomatic abstract set theory.

The development of this Strand benefits as materially as do other Strands from a spiralling escalation of material wherein old concepts are rediscovered along with newer ones. For Kindergarten and the first grade it is probably sufficient to introduce the notion of a set as a collection of things; allow the pupils to identify natural examples of sets and to form new sets from old by joining (set union) and intersecting. Sets of objects can be indicated by many devices such as placing cutouts on a flannel board or drawing pictures of the members of the set. If several sets are to be pictured together, then they can be separated by surrounding each set with a circle-like curve. At this introductory level, the teacher will use simple terms correctly, encourage their use by her pupils, but not require a mastery of this vocabulary from the students.

Later, in the intermediate grade, the problem of how to denote sets will need a more sophisticated device than a picture. We recommend the convention of listing the elements of a set, the members of the set, between braces. Thus, the set of the first 4 positive odd integers is denoted $\{1, 3, 5, 7\}$. It is important to point out that the order of the listing, or the number of times an element appears in the list does not affect the set. Thus, $\{1, 3, 5, 7\} = \{7, 1, 5, 3\} = \{3, 5, 7, 5, 1, 3\}$. At the same time, examples should be presented of large sets where it is inconvenient or impossible to list all the elements. For example, the set of all positive odd integers less than $\sqrt{1049}$ or the set of all positive odd integers.

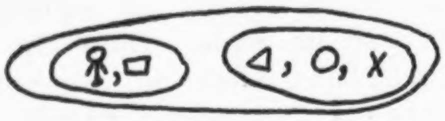
Arithmetic, measurement, and geometry at all grade levels require extensive use of the basic concept of one-to-one correspondence, commonly called matching. This concept relies implicitly, if not explicitly, on a preconceived notion of set to describe what is being matched. The abstract nature of the number four can be learned by considering many different sets and by noting that the elements of some sets can be matched in a one-to-one correspondence with the elements of a particular set, { ○, △ , ∅ , ▱ }, to which we assign the number four.

In presenting arithmetic and geometry at an intermediate or advanced level, the teacher should use set operations and correct symbolism: set union (∪), set intersection (∩), and set difference (-). We shall give some examples of their use. In the first two we suggest an elementary equivalent.

- in addition (set union)

$$\begin{array}{ccccc} 2 & + & 3 & = & 5 \\ \{x,y\} & \cup & \{a,b,c\} & = & \{x,y,a,b,c\} \end{array}$$

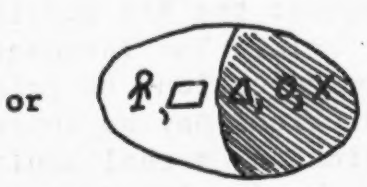
or



- in subtraction (set difference)

$$\begin{array}{ccccc} 5 & - & 3 & = & 2 \\ \{a,b,c,x,y\} & - & \{a,x,y\} & = & \{b,c\} \end{array}$$

or



- in solving equations (solution set)

  (a) Solve for □: □ + 5 = 7. The solution set is {2}.

  (b) Solve for x: $2x + 1 = 21$. The solution set is {10}.

- in solving inequalities (solution set)

  (a) What whole numbers satisfy 4 < □ ≤ 8?

  The solution set is {5,6,7,8}

  (b) What real numbers satisfy $2x^2 + 5 < 7$?

  The solution set is $\{x: \quad |x| < 1\}$

- in addition and subtraction of fractions

$$\frac{1}{18} + \frac{1}{30} = ?$$

(a) Solution using least common multiple

The set of multiples of 18
$= A = \{18, 36, 54, 72, 90, 108, 126, 144, 162, 180, \ldots$

The set of multiples of 30
$= B = \{30, 60, 90, 120, 150, 180, \ldots\}$

The set of common multiples of 18 and 30
$= A \cap B = \{ 90, 180, \ldots\}$

The least common multiple of 18 and 30 is 90.

Hence $\frac{1}{18} + \frac{1}{30} = \frac{5}{90} + \frac{3}{90} = \frac{8}{90}$

(b) Solution using greatest common divisor

The set of divisors of 18 = C = $\{1,2,3,6,9,18\}$

The set of divisors of 30 = D = $\{1,2,3,5,6,10,15,30\}$

The set of common divisors of 18 and 30 = $C \cap D = \{1,2,3,6\}$

The greatest common divisor of 18 and 30 is 6

Hence $\frac{1}{18} + \frac{1}{30} = \frac{1}{6} \left(\frac{1}{3} + \frac{1}{5} \right) = \frac{1}{6} \times \frac{8}{15} = \frac{8}{90}$

- in geometry (set intersection)

Two different straight lines intersect in at most one point.

Figure 1

- in geometry (one-to-one correspondence)

Two line segments have the same number of points. (See Figure 2)

Proof. Let line segments $\overline{AD}$ and $\overline{BC}$ intersect at O. The point P on AB corresponds to the point Q of $\overline{CD}$ which is the intersection of the line determined by the two points P and O and $\overline{CD}$.

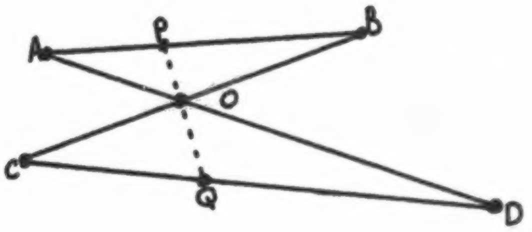


Figure 2

The concept of a subset of a set is equally natural. The set A is said to be a subset of the set B if and only if every element of A is an element of B. For example, "The set of even integers is a subset of the set of all integers", "A line segment is a subset of a line", or "The solution set of $-1 \leq x - 1 \leq 0$ is a subset of the solution set of $0 \leq x^2 \leq 1$." The notation $A \leq B$ may be used to denote that A is a subset of B.

For multiplication and the number plane the notion of an ordered pair of elements is important. For example, how is multiplication involved in answering the problem

> "If I have 3 shirts and 4 pairs of slacks how many different combinations can I wear?"

We need to count the number of sets (shirts, slacks). This set is really ordered by the nature of the garment: shirt first, slacks second. The choice of which is first and which is second is not important, but it is important not to confuse which is which!

In mathematics, we do pay attention to the order. An ordered pair is a sequence consisting of two terms listed in order. Thus the ordered pair (1,2) is different from the ordered pair (2,1). Indeed these two ordered pairs are plotted as distinct points (see Figure 3) on the number plane. An ordered pair has a first element, a, and a second element, b. Thus, the ordered pair (a,b) is distinct from the ordered pair (b,a), unless a = b. Ordered pairs are the means by which a function is carefully defined.

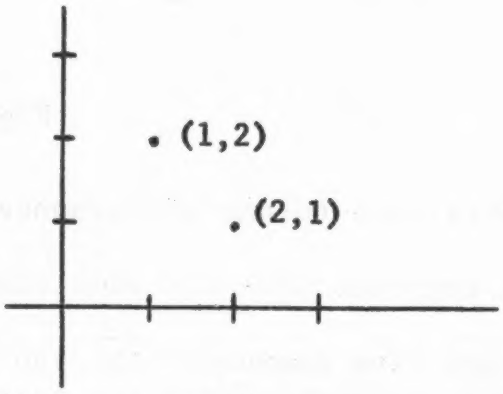


Figure 3

By grades 7 and 8 pupils may be expected to use correctly the elementary terms for set concepts. At this level a learner will probably need a better way of expressing sets. A way which has proved to be extremely useful is the set builder notation. For example, the solution set of $2x + 3 < (2+5x) - 8$ was the set of all numbers greater than 3. In the set builder notation we would write this as

$$\{x:\ 2x + 3 < (2+5x) - 8\} = \{x:\ x > 3\}$$

and we would read the latter as we have just indicated:

> "$\{x:\ x > 3\}$ is the set of all numbers x such that x is greater than 3".

Using this scheme we can write the set of all positive odd integers less than $\sqrt{1049}$ in various ways, such as

(1) $\{x\colon\ 0 < x < \sqrt{1049}$ and x is an odd integer$\}$.
(2) $\{x\colon\ 0 < x,\ x^2 < 1049$, and x is an odd integer$\}$.
(3) $\{x\colon\ x = 2y + 1$ for some integer y and $0 < x < \sqrt{1049}\}$.
(4) $\{2y + 1 : 0 \leq y,\ (2y+1) < \sqrt{1049}$, and y is an integer$\}$.

In general we enclose in braces { , }, a symbol for a typical element of the set (a "John Doe") followed by a colon which stands for "such that", followed by a list of conditions which the element must satisfy. In this vein we could express the set of possible candidates for President of the United States in 1968 as {x: x is a native born citizen of U.S.A. and x is at least 35 years old}.

In (4) above we gave, instead of x, an expression (2y+1) for an element in the set and then described the condition on the symbol y appearing in the expression. As a final example we define the set of Pythagorean triples: $\{(a,b,c)\quad 0 < a \leq b \leq c,\ a^2 + b^2 = c^2\ ,\ a,b,c$ integers$\}$.

In the final developmental stage of the language of sets in Kindergarten and grades one through eight, we expect the pupil to use these set concepts, terms, and notations, correctly and with reasonable precision. These will play an important role as he studies inequalities, solution sets, informal geometry, functions, probability and statistics, and the applications of mathematics.

Strand 7. Functions and Graphs

Most of the mathematics is concerned with relations. The young child learns at an early age to relate certain objects or sounds with other objects. For example, he relates a child with the child's parent. Intuitively, we may think of these associations as recognition of certain ordered pairs of objects. Thus the child learns pairs of the form (name, object) in discovering language. Sets of related pairs of objects are studied throughout the elementary mathematics program. They are often described by graphs and tables, and we describe the coordinate plane of geometry in these terms. These notions lead directly to the concept of function which permeates all of mathematics and science. This concept should be developed, named, and used in the elementary school program.

The process of relating certain pairs occurs early in the mathematics program. In the beginnings of arithmetic, the pupil learns to associate each set of objects with a number and to count by pointing to the objects in sequence and pairing them with the set of ordered number words. He learns that counting is a way of determining what number is to be associated with a certain collection of objects; thus, counting determines certain related pairs (set, number). He also becomes aware that different sets of objects may be paired with the same number but that a finite set of objects is paired with one and only one number.

Another early example is furnished by the relationship of "greater than". We may relate a member to the numbers which are greater, and think of the ordered pairs of the form (member, greater number). This is an example of a relation in which a single number is related to many other numbers.

A pictorial description of relations between certain pairs of objects occurs early in the mathematics program. Plotting and graphing are the ways we make pictures of relations. Plotting may be initiated with games, such as the tic-tac-toe game. A class may graph the height of a plant on successive days of a month, or the temperature may be recorded for each day. The experience of such graphing reinforces the concept of the number line,--both vertical and horizontal, presents a picture for linear relationships and provides an excellent way of grasping relationships intuitively. The method is also invaluable in application of mathematics. A class may record the length of a spring (rubber band) as weights of successive sizes are suspended, or, at a more advanced stage, the period of a simple pendulum as the length of the string is varied. In the teaching of measurement a class may record the number of ounces (or any standard unit) of water required to fill cylindrical jars of various diameters to a fixed depth. Or in the study of geometry, a class may plot circumference as related to diameter for different circles.

There is also learning of the reciprocal sort. We describe a relation between numbers by plotting related pairs on the plane; we also describe the plane in terms of all ordered pairs of numbers. This connection (point in the plane with ordered pair of numbers) is basic to the mathematics which connects geometry and algebra. It is also basic to the understanding of maps and, more generally, scale drawings. The elementary program should include the study of the coordinate plane.

There is one sort of relation which is of particular importance to science and mathematics. This relation is such that for each member of a class of objects there is just one object to which it is related. For example, for each child there corresponds just one natural mother. Such a relationship is called a functional relationship, or just a function. Thus the related pairs that form a functional relationship have the property that there is, at most, one related pair with a given first member. It should be emphasized that order is most important. As indicated above, the relationship in which each child corresponds to his mother is a function, thus the set of ordered pairs of the form (child, mother) is a function. However, the relationship in which each mother corresponds to each of her children is not, thus the set of ordered pairs of the form (mother, child) is not a function since a single mother may have several children.

There is another conceptual description of function with which the students should become familiar. Since a function relates one object to a unique second object, we may think of it as a sort of machine, as in Figure 1:

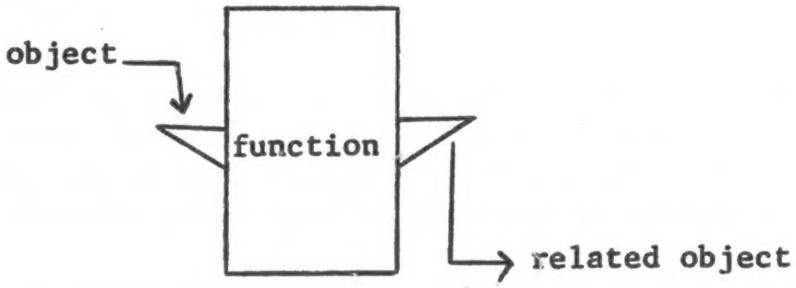


Figure 1

Figure 2 shows how this picture might be used with the squaring function in which every number corresponds to its square.

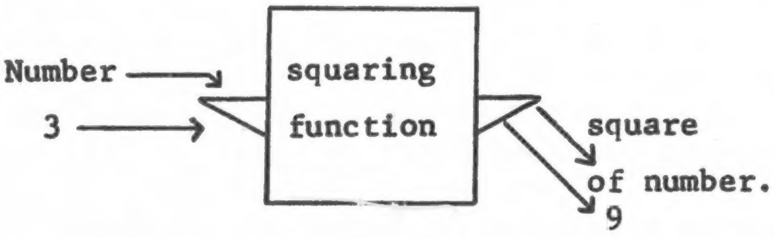


Figure 2

Figure 3 illustrates the counting function.

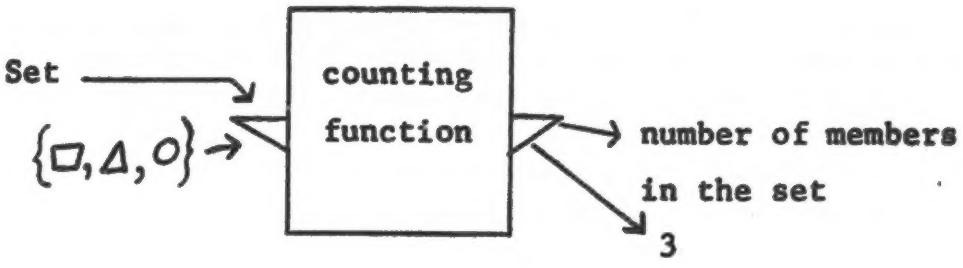


Figure 3

Functional notation should be introduced, and be used systematically by the end of the eighth grade. Several different notational schemes may be suggested. It is important to use different notational schemes upon occasion since different notations are suggestive of different aspects of the function concepts. We illustrate here some of the notations possible for the squaring function.

$$S(2) = 4, \quad S(3) = 9, \quad S(4) = 16$$

or

$$S: \begin{array}{l} 2 \rightarrow 4, \\ 3 \rightarrow 9 \\ 4 \rightarrow 16 \end{array}$$

or as a table,

Δ	$S(\Delta)$
2	4
3	9
4	16

The description, or formula: $S(n) = n^2$, for each number n, is also useful and appropriate.

The function concept includes mathematical operations. Pupils have encountered binary operations in learning the number facts although the terminology of binary operation is probably new to them. The basic multiplication facts, for example, describe this binary operation on the set of whole numbers. The operation is binary because only two whole numbers are involved. Thus a binary operation on numbers, such as addition or multiplication, is a special way of relating two numbers with one number. The function may be presented in a table, such as the familiar addition or multiplication table. In multiplication, the number pair (3,4) is paired with the number 12; the number pairs (4,3), (2,6), (6,2) (12,1), and (1,12) are also paired with the number 12 in the multiplication operation. This indicates that in a binary operation many different number pairs may be paired with the same single number but that a given number pair is associated with only one number.

It is important that pupils have the previously outlined experiences before they have completed the eighth grade. Intuitive experiences will pave the way toward conceiving a function as a set of ordered pairs in which no two pairs have the same first element. The pupil should realize that functions can be presented or described by statements, formulas or equations, tabulated data, and graphs. The pupil is then on his way toward mastering an important concept which is the key to understanding many mathematical ideas that have far-reaching applications. The mathematics program should make the student familiar with the notations for a function listed above and the student should be able to plot linear and quadratic functions and functions such as the greatest integer functions.

Strand 8. Logical Thinking

In many walks of life, we are concerned with the organization of thoughts so that we can work out other problems or bring meaning to the ones we are working on. We make decisions based upon conditional statements. New endeavors are often determined by conjecturing what would happen "if" we did it another way or changed the rules by which we work. The beginnings of such logical thought or reasoning ability have their roots in the elementary school program. This is especially true in the elementary school mathematics program where the emphasis is on organizing ideas to clarify the meaning of what is learned and to help pupils learn to think for themselves. Logical thinking at the elementary school level does not imply formal proofs or a study of logic per se. Logical thinking and deductive reasoning at the elementary level is a matter of well-organized common sense. Common sense can be sharpened by the use of standard logical techniques such as Venn diagrams and truth tables. Children should be able to decide whether a particular mathematical construct fits a definition and to recognize a specific instance of a general principle. The following examples will serve as a means of illustrating the types of logical thinking that can be started in the elementary school.

1. The Venn diagram in Figure 1 shows that all positive odd numbers are whole numbers and that there exists a (at least one) whole number which is not odd.

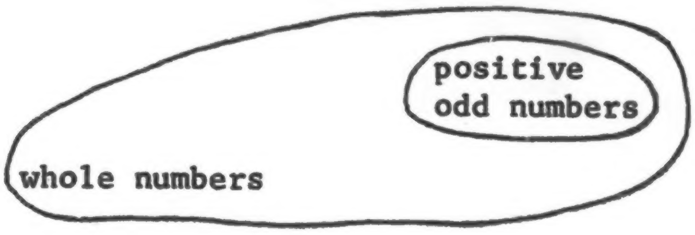


Figure 1

An important aspect of this example is the study of the quantifiers "all" and "there exists".

2. Geometric models for numbers can be used to make generalizations and inferences about odd and even numbers.

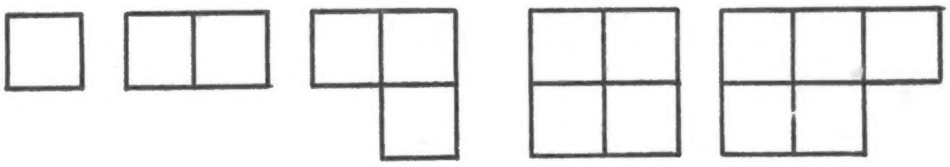


Figure 2

The even numbers are represented by rectangles and the odd numbers, greater than 1, are represented by rectangles with a square tacked on. By fitting the pieces together, pupils may discover and conclude that (a) the sum of two odd numbers is an even number, (b) the sum of an odd number and an even number is an odd number, (c) the sum of two even numbers is an even number. Later on, pupils see that $2n$ and $2n + 1$ are ways of designating odd and even numbers. In this example, the model is used to justify the statements (conclusions or generalizations) that are made. A variation of this technique can be used to study how odd and even numbers behave under multiplication.

3. A and B are two points on a number line. B represents a number greater than A. Point H is located between points A and B and is closer to point B than it is to point A. What statements can you make about the numbers represented by points A, B, and H ?

4. Suppose that the following facts are told us about three numbers:

   i) Each number is a positive integer.
   ii) The sum of the three numbers is an odd number less than 12.
   iii) One number is three times another.
   iv) The product of any two of them is an odd number.

   Which of the following statements are consistent with the given information? Of these, which are consistent with the first statement? Of the consistent ones which, together with the preceding ones, determine the numbers uniquely?

   1) The numbers are all equal.
   2) Two of the numbers are odd. The other is even.
   3) The sum of two of the numbers is one less than the third number.
   4) The numbers are all different.

5. Suppose there are 33 members in a class, and that each pupil either rides a bicycle to school or brings his lunch or both. If 21 ride bicycles and 15 bring their lunch how many do both? (The Venn diagram of Figure 2 helps the analysis.)

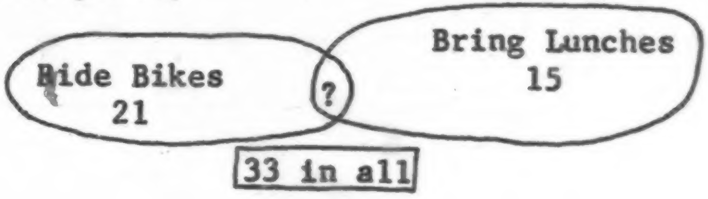


Figure 3

At the elementary school level, the study of logical thinking follows the meaningful buildup of principles and generalizations. In most instances, these generalizations are the outgrowth of the critical analysis of many related problems in which a common element or happening occurs. These generalizations or definitions become the starting point for the study of logical thinking in mathematics. At the elementary level, our use of many of these mathematical terms are of a non-technical nature. For instance, the term less than can be introduced by showing that two is less than five through an attempt to establish a one-to-one correspondence between the objects in a set of two objects and the objects in a set of five objects. Next, the less than symbol, (<) can be introduced and described in operational terms. For example, $2 < 5$ means that two is named in the counting sequence before five. A start toward a mathematical definition is made when it is understood that $2 < 5$ means that there is a counting number that can be added to two to get five. The formal definition can wait until more precise algebraic terminology is available.

The use of variables is another example of the non-technical use of a symbol in mathematics. In the elementary school mathematics program, the symbols □ , ◇, n, x, ?, ○, etc., are used in a mathematical sentence to indicate that something is unspecified in that sentence. There is something that we want to find out from the information that has been given us. When we do find it out, we may write it in place of the variable or symbol. The missing element may be a numeral or an operational sign. In most instances it is viewed as standing for some undetermined number. Later on it takes on the meaning of representing any of several numbers that can be used in the given sentence. In elementary mathematics, a variable is just a symbol which appears in a mathematical sentence to stand for a mathematical object (element, number, point, line, function; depending on the context) as yet unspecified or to be determined. Such a symbol is just a generic name, like "Mr. John Q. Public", for a typical member of the set under consideration. The mathematical sentence of which a variable is a part places conditions or gives properties which may, of course, determine it uniquely or show how all members of the set behave under these conditions. (We recommend that no definition for the term "variable" be attempted and indeed we suggest that the term be avoided altogether.)

Logical thinking may also be based upon the definitions or agreements that are given to some of the words used in mathematics. In the early grades, pupils become acquainted with the use of the words all and some, which are known as quantifiers in logic. The pupils are able to distinguish between the meanings of the two statements, "all balls are red" and "some balls are red". In mathematics, the word some means "there is at least one".

Terms such as and, or, if-then, and not are important logical terms and should be informally introduced in the elementary grades. In order to see why it is important in mathematics for one to agree on the meaning of these terms, one can examine the multiple meanings of each of these words as they are used in the English language. Pupils need to distinguish between "numbers that are both odd and prime" and "numbers that are either odd or prime" and to recognize that "or" does not exclude "and". The meaning of the if-then sentence can be analyzed in simple situations such as this: "If Helen goes to the movies, then Jim goes," noting that this sentence would be false only if Helen went to the movies and Jim did not go. The if-then logic sequence can be further extended in the upper grades to geometric applications such as in testing the accuracy of the statement, "If a plane figure has four congruent sides, then it is a square."

Everyone should come to realize that mathematics is concerned with establishing the truth or falsity of sentences like "If A then B." It is only when such a sentence is used that the pupil is concerned with the truth of A. The main deductive scheme of logic permits us to deduce B from knowing the truth of both "If A then B" and "A".

It is very important for each pupil to learn how to state the negation of a statment. For example, the negation of "All isosceles triangles are equilateral" is "There exists a triangle which is isosceles but is not equilateral." The negation of "There is a largest whole number" is "For every whole number there is a whole number which is greater than it."

Elementary school children seem to enjoy simple syllogistic arguments. These arguments may be given in humanistic situations such as

Pretty girls are elected "Queen of the May."
Lana is this year's "Queen of the May."
Therefore, Lana is a pretty girl.

These examples should be given in conjunction with mathematical examples such as

Prime numbers larger than 3 are odd numbers.
Forty-one is a prime number.
Therefore, forty-one is an odd number.

In addition, these arguments should be accompanied by the appropriate Venn diagrams. See, for example, Figure 3.

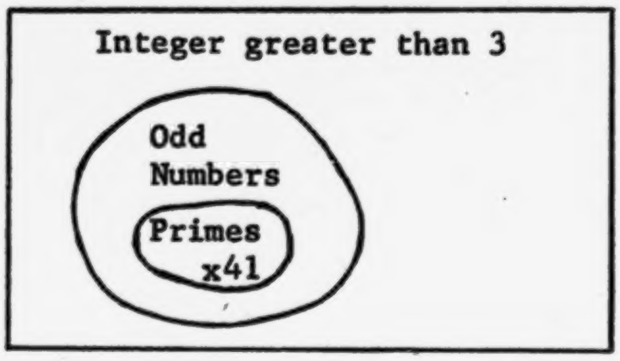


Figure 3

Logic is also concerned with developing other types of patterns for drawing valid inferences. Simple inference schemes should be informally introduced and kept at a level that makes sense to pupils. For example, what conclusion or conclusions could be drawn from each of these situations?

1. John is the tallest boy in his class. John and Pete are in the same class.
2. Suppose three pieces of chalk were placed into two empty containers while you were not looking.
3. If $\frac{2}{3}$ of a number is 6, then $\frac{1}{3}$ of the number will be $\frac{1}{2}$ of 6 or 3. What is the number? (Many other deductive reasoning situations similar to this one occur in the arithmetic program.)

The development of logical thinking is an integral part of the instruction in all phases of the elementary mathematics program. At this level a separate and distinct treatment of logic or logical thinking is not intended. The teacher's editions of the textbooks should point out specific situations in which the teacher can extend the lesson beyond the mere acquisition of facts and skills into the level of logical thinking.

Strand 9. Problem Solving

Mathematics exists to pose and to solve problems--all kinds! Problems come from the "real world" and from the "ideal world" of mathematics. Some problems have profound implications, others are merely riddles; some problems have "solutions," some only lead to other problems. Here we take the long view that a "problem" is an articulation of something that requires analysis before understanding. We realize that this criterion varies from person to person. Because analysis is required and because it is personal, understanding does not come without some burst of invention within the mind. How can we facilitate this creative mental process for elementary school children?

In this Strand we attempt to make explicit some methods for this facilitation. It is important to differentiate this Strand from that of Strand 4, Applications of Mathematics. In that Strand we were concerned with problems arising outside the domain of mathematics; indeed, the key process in the application of mathematics is the construction of a mathematical model or formulation of the problem. In this Strand we shall be concerned with the solution of mathematical problems with mathematical methods or with physical models which reflect the mathematical situation. The distinction we make is a real one. Almost all the problems we have cited as examples in this report have not been applications of mathematics outside itself. We need more of those and that is why we have emphasized Strand 4. But we must also solve mathematical problems. That is why we now emphasize Strand 9.

The ability to formulate meaningful problems has more value in the market place than just the ability to solve problems. As we have indicated so often before, each attempt on the part of a pupil to formulate a problem should be encouraged. A high point of any teacher's career comes when his pupil asks a question the teacher has not previously considered. Of course it is a subtle task to be sure that a question has been reasonably formulated. We shall attempt no definition of a "reasonably formulated" question; certainly pupils should be given considerable experience in formulating questions. We must not insist that all irrelevant data have been trimmed away; indeed, until a solution is obtained exactly what is relevant is unknown. Even data judged irrelevant can suddenly lead to a new insight or method in solving the problem itself.

It is our belief that most elementary textbooks do not contain real problems. They do contain "verbal" problems. Often these are thinly disguised computational exercises. For example, "John has 4 marbles. Dick as 6 marbles. How many do they have together?" Exercises of this type are boring for the learner who can read. The difficulty a slow reader has may have little to do with mathematical concepts. We are hopeful that more problems will be included which require pupils to explore, analyze and investigate. We strongly recommend that at regular intervals problems occur which are open-ended in the sense that they invite conjectures and generalizations. We shall include several examples of this type of problem at the end of this Strand.

One of the basic objectives of the mathematics program should be to provide pupils with opportunities for problem solving and to assist them in devising means for attacking these problems with an expectation of success. In this Strand we shall distinguish between problem solving strategy and problem solving tactics. By strategy we mean a general overall plan of attack. By a tactic we mean a single technique which will help with a part of problem. This distinction is, of course, a relative matter; a strategy in one grade may well be a tactic in another. For example, the strategy of regrouping for easy addition which might be taught in the early grades becomes a tactic in the upper grades when it is used to simplify mathematical sentences involved in a more complex mathematical analysis. We begin with strategy.

We recommend that the strategic principles we now present be expanded in the teacher's manual so as to be introduced to each pupil. We insist that these not be made a format to be followed or become a "step-by-step" procedure to which all solutions must conform. We emphasize that in problem solving a creative method of solution is more valuable than a routine answer. At the same time, if a pupil has flashed to a correct answer his thought process should not be hamstrung by having to conform to unnecessary formalism.

A first step in any strategy is to understand the problem. Regardless of the origin of the problem the solver must understand the problem so well that he can restate it in his own words. He should be able to identify given data or the hypotheses and pinpoint the object of the problem. At this stage it is sometimes useful to guess an answer and try it out.

This first step in attacking a problem is one which has proved to be the toughest for the beginner. It is upon the first reading of the problem that despair sets in. Here the teacher must begin to build confidence. Anything the pupil says can be helpful; even repeating verbatim the statement! Every pupil should be encouraged that any comment, even the most prosaic or trivial is a right step in the right direction. Indeed, we find that no comment is trivial or prosiac; each gives the teacher feedback as to how the pupil has heard the problem. With such information a teacher can better help his pupil. The teacher can stimulate discussion by suggesting information, techniques, experiences, skills, or procedures that are already known that might help the solver in this particular problem.

There are several tactics available to the solver at this point. He may construct a diagram or picture of what is required. (Children in the early grades are fond of inserting many superfluous details in such pictures. These should be tolerated for they help to point out what a bare bones skeleton the mathematical model really is.) He may construct a physical model or use materials to simulate the problem. The use of aids should be encouraged. As practice and skill increase the pupil should be led to the creation of models which concisely present the essence of the problem. The pupil may also find the construction of a graph a handy medium to describe various features of the problem.

__Example__. Janet has 7 cents. She goes to the store to buy one candy cane that cost 1 cent and as many jumbo packages of bubble gum as she can. If each package of bubble gum costs 2 cents how many can she buy?

Figure 1 suggests a picture which a pupil might draw to aid his thinking.

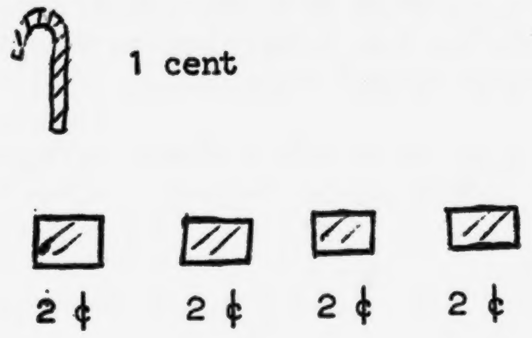


Figure 1

The tactic of making a picture of any mathematical situation might well become the habitually accepted thing to do. This habit can be established in pupils by imitation of the teacher who will always do this or often suggest to the class that they do it together. The tactic should be started with the first mathematical experiences when pupils are learning concepts and skills and be carried on to more complex problems where the diagrams and graphs will be more complex.

A good tactic at this point is trial and error, or the informed guess. The problem about Janet might be further analyzed by a series of pictures, as shown in Figure 2.

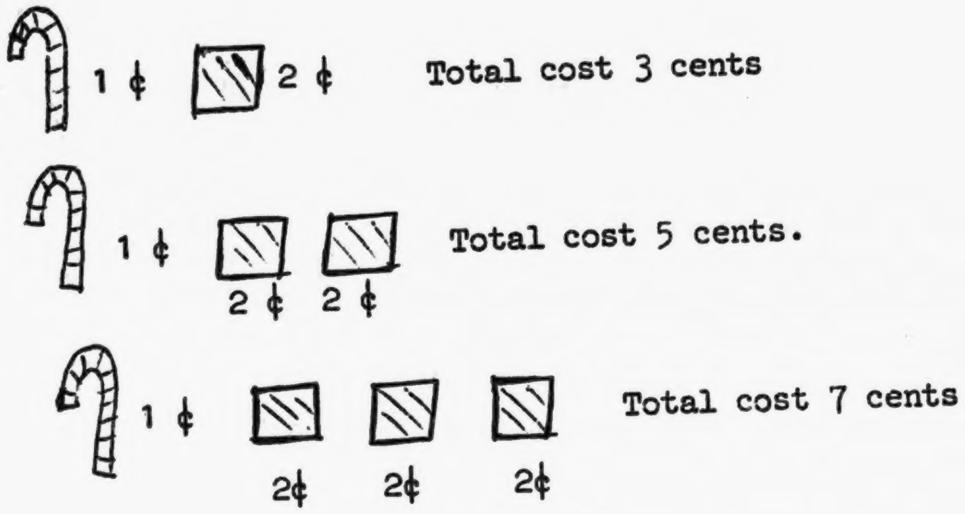


Figure 2

Once the problem is understood a next step might be to translate the statements of the problem into appropriate mathematical language and symbolism. This step is a crucial one, for in it, the conditions of the problem must be made explicit and correct. This step in problem solving is essentially that of constructing a mathematical model of the problem. The process of model building was discussed in the Strand, Applications of Mathematics.

The preceding eight strands provide tools for this part of problem solving. Each of the strands has something to say about a language for expressing and relating mathematical concepts.

A statement expressing relations between mathematical concepts is called a mathematical sentence. Such a sentence often consists principally of symbols.

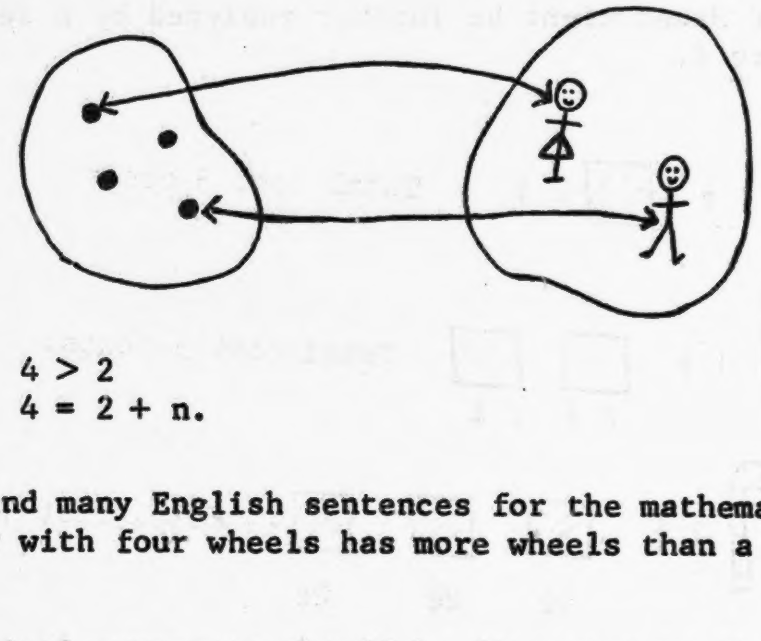
Examples: 1. Solve $2\square + 1 = 7$
2. Solve $2\square + 1 = \triangle$
3. Angle ABC is a right angle since $AB \perp CD$.
4. Area (of a rectangle) $= L \cdot W$
5. If $x^2 \leq 4$, then $-2 \leq x \leq 2$.

One of the tactics we can help children master is the translation of English sentences into mathematical sentences and conversely, the interpretation of mathematical sentences as English sentences. This skill begins as soon as any mathematical experience.

Example: "If Billy has four marbles and Jane has two dolls,
then Billy has more marbles than Jane has dolls."

This could be translated variously as:

1.



2. $4 > 2$
3. $4 = 2 + n$.

Conversely, we can find many English sentences for the mathematical one, $4 > 2$. One is "A car with four wheels has more wheels than a bike with only two wheels."

Complex mathematical sentences should be dissected so that the student comes to learn what each part of the sentence means. Such discussions will point out the different role played by symbols indicating operations or relations ($+, -, \cdot, \perp, <, \leq, \div, \not<$) and symbols which denote numbers or points or lines or symbols which stand for these (x, a, □ , △ , ◇ , P). These discusssions will also point the need for the use of parentheses and the role of the associative law. For example, 6 - (3 - 2) should be compared with (6 - 3) - 2. The notion of equivalent mathematical sentences will arise naturally when free rein is given to pupils to construct mathematical statements corresponding to English ones. Introduce the notion of a solution set.

Mathematical sentences may assist a pupil to make generalizations in the early stages of his mathematical education. If he observes that $0 + 1 = 1$; $0 + 2 = 2$; $0 + 3 = 3$; and so on, he should be able to generalize, that for every whole number n, $0 + n = n$.

A single mathematical sentence will often unify different problem situations. For example, the so-called three cases of percent can be united in the single sentence

$$\frac{a}{100} = \frac{c}{b}$$

which relates the percent, a, the base, b, and the percentage, c.

Thinking through a problem situation may result in the formulation of a mathematical sentence. When a pupil learns to describe a problem in his own words and then to state the relationships by use of a mathematical model in sentence form, he has a sound approach to problem solving. The process of building a mathematical model should not be mechanical but should lead to a deep understanding of the mathematical process. Pupils who understand and use this approach to problem solving seldom ask, "Do I multiply or divide?" Procedural steps in problem solving should be established by the individual pupil through his ability to verbalize what he did in any given situation, rather than by a prescribed set of steps to follow. Allow him to think for himself, permit him to get into situations that result in failure, ask him questions that redirect his thinking, and eventually he will attain a solution. Ask the solver to describe the situation to others. The teacher is always there to listen!

It is important to notice that diffent situations lead to the same mathematical sentence. These translation projects provide good opportunities for the class and the teacher to think out loud together. The problem about Janet and her seven cents leads to the mathematical sentence

$$2x + 1 = 7$$

Here are three more problems which lead to the same sentence.

1. Hale High has played 7 football games. One game ended in 6-6 tie. If Hale won half of the other games, how many did they win?
2. Jim lives 7 blocks from school. Jim meets Bill one block from school. When Jim met Bill, Jim had walked twice as far as Bill. How far does Bill live from school?

3. There are 7 children in the Hardy family. Tom is the only boy. Every other child is a girl and every girl has a twin sister. How many pairs of twins are there in the Hardy family.

There are of course many equivalent ways to formulate a mathematical sentence associated with a specific problem. Here are some ways any of which could have been an equally natural way to formulate the mathematical sentence associated with the above problems

$$7 = 1 + x + x$$

$$7 - 1 = 2x$$

$$x = \frac{7 - 1}{2}$$

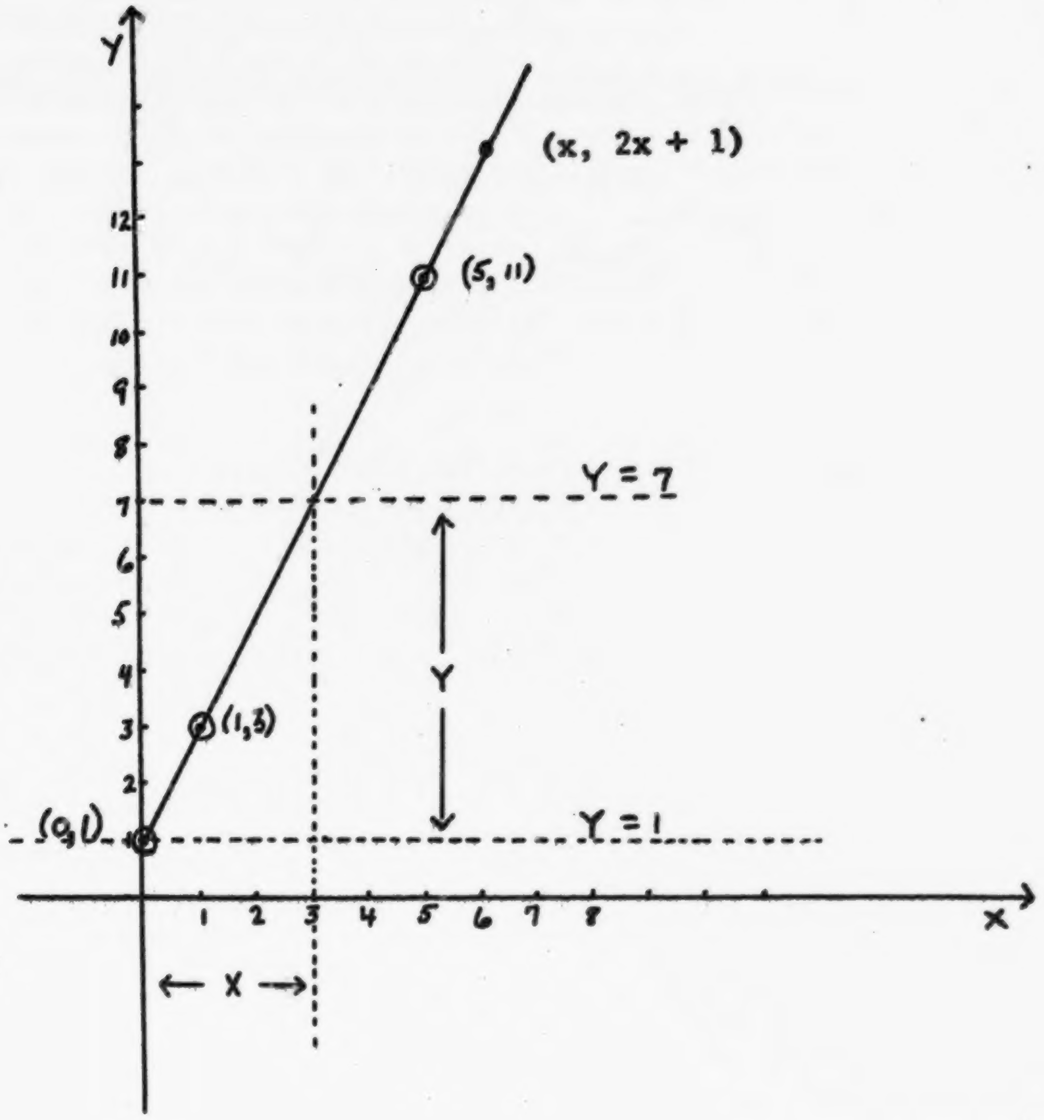
$$\left\{\begin{array}{l} 7 - 1 = 6 > 2 \\ 7 - 1 = 6 > 2 + 2 \\ 7 - 1 = 6 = 2 + 2 + 2 \end{array}\right\}$$

This last system of sentences corresponds to the trial and error method of solving the candy cane-bubble gum version of the problem as worked verbally by a fourth grader.

A later step in a strategy may concern the mathematical analysis of the mathematical model or the mathematical sentences which express the hypotheses of the problems. This analysis is of course the reason for having previously introduced various techniques and acquired special skills. On the other hand the need for this analysis may provide the motivation for the discussion of new mathematical principles. Some tactics which may be helpful include reformulating the conditions to be analyzed, studying associated functions, considering extreme cases for any of the parameters involved, or creating new aids such as graphs, models, or pictures.

Example: Suppose the mathematical condition required a solution for $2x + 1 = 7$. We could

(a) Try various numbers for x.
(b) Solve a similar, but easier, equation. Replace $2x$ by y and solve $y + 1 = 7$. Finally note that $2x = y$ or $x = \frac{1}{2}y$.
(c) Write an equivalent sentence: $2x = 6$.
(d) Make a graph (Figure 3) of $2x + 1 = y$ from the information obtained in (a)
(e) Change the sentence to $0 \leq 2x + 1 \leq 7$. Do this from the graph in (d).
(f) Change the sentence to $0 \leq 2x + 1 \leq 7$. Do graphically.
(g) Change the sentence to $2x = 7$ or $2x - 1 = 7$. What are the effects of these changes on the graph shown in (d)?



Information we can read from the graph:

If $1 \leq 2x + 1 \leq$ then

$0 \leq x \leq 3$

Figure 3

While not all of these tactics make the problem easier they do give familiarity with the sentence! A pupil who has seen, and created, the ins and outs of routine problems like these will have little difficulty with more complicated mathematical sentences.

A final part of a strategy is the interpretation of the answer. An answer must seem reasonable! A part of this strategy is also "checking your answer." It is important to differentiate between a check carried out for purposes of checking the accuracy of the problem analysis and the computational work, and a check because some logical step in either the model building itself or in the analysis (e.g. the introduction of extraneous roots) requires it. An accuracy check must be carried out because of a personal desire on the part of the solver (or his teacher) while a logical check is a part of the problem itself.

In summary, we emphasize that in all problem situations the pupils must be furnished the basic skills necessary to make the kinds of analyses which will lead to a profitable attack. Teacher direction should help the pupils to recognize that a problem exists. They will find solutions with a minimum of interference in their experimentation. Necessary arithmetic skills can be given as they are needed, but information for analysis is pupil work. Visualization of problems, and sometimes the solutions themselves, should be stressed by the use of graphs, flow charts, and any other visual aids they may devise. Conditional statements should be used to a much greater degree than they have been in the past. The logical developments to be expected by self-questioning should be investigated and used. Where number manipulation is required it should be done carefully and the result investigated. In most cases numerical results should be anticipated by estimating in advance. This technique should be introduced early in the mathematical experience of all pupils, and should become second nature to them. Possibly situations will arise, and they definitely can be contrived, where an estimate will point out an error in analysis or translation before any numerical manipulation is started. Results should never be accepted without testing. Many times the post discussion will be as valuable as the preliminary approach, and opportunities for teaching from such discussion should not be overlooked.

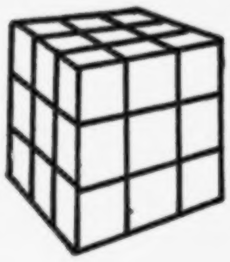
In the course of the discussions both at the time of problem analysis and during solution analysis, the teacher must be receptive to all pupil suggestions and comments. Rejection at any time may defeat the purpose of the discussion, at least for the individual pupil, and could lead to abandonment of the particular problem, and possibly a regression in the whole problem solving technique. Pupils' self image must be maintained at a level acceptable both to them and to the group if discussion is to be kept profitable.

Examples of Open-ended Problems:

1. Heidi said she had a good idea for a problem. Make a 3 by 3 by 3 cube and paint the surface area. Now find out how many of the 1 by 1 by 1 cubes are painted on one side, on two sides, and so on.

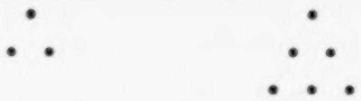
   Complete the chart below:

Number of 1 by 1 by 1 cubes painted on						Volume
0 Sides	1 Side	2 Sides	3 Sides	4 Sides	More than 4 sides	Number of 1 by 1 by 1 cube

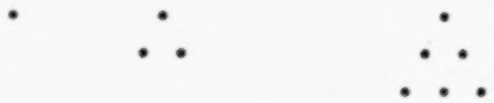


Solution: 1 6 12 8 0 0 27

2. Three and six are triangle numbers, as they can be shown in triangular form:



   The first three triangular numbers are:

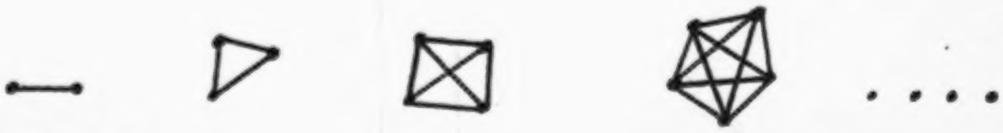


1, 3, 6,

   Find the first 10 triangular numbers.

   The triangle numbers are found in many patterns. For example, the number of segments connecting 2, 3, 4, 5, ... points.

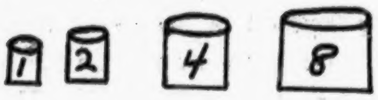
Non linear points



NOTE: There are many similar patterns including square and pentagonal numbers.

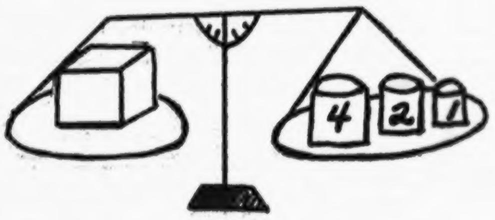
3. Richard has a balance and a set of weights.

1, 2, 4, and 8 pound weights.



In the diagram at the right show that the box weighs 7 pounds.

The equation for this is $7 = 4 + 2 + 1$



Write the equations which show how boxes of 1, 2, 3, 4, ..., 14, 15 pounds can be weighed using the balance.

Another interesting problem is to use pound weights of 1, 3, 9, 27, ... This problem involves base 2 and base 3 respectively.

4. Jane has 17 coins which total $1.00. What are they?

Solution: (a) 2 quarters, 3 dimes, 2 nickels, 10 pennies.
(b) 1 half dollar, 1 quarter, 3 nickels, 10 pennies.

5. Bill has 16¢. What coins may he have?

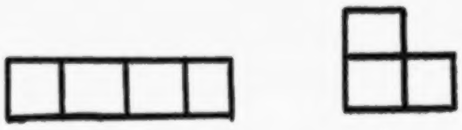
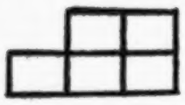
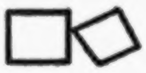
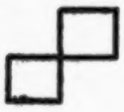
Complete the chart

⑩¢ 10¢	5¢	1¢
1	1	1

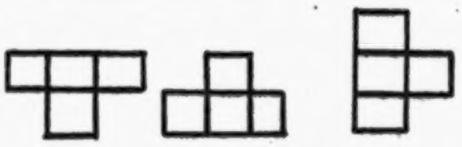
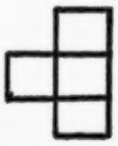
Solution:

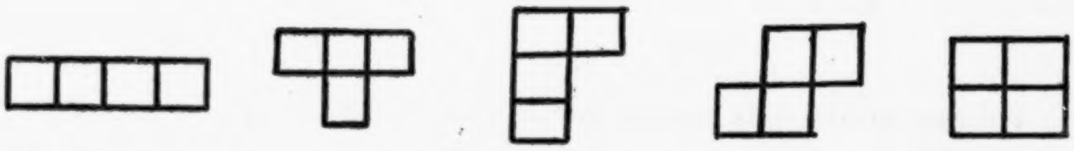
D	N	P
1	1	1
1	0	6
0	3	1
0	2	6
0	1	11
0	0	16

6. Connected Sets of Squares

 and  are connected sets of squares.  and  are not connected sets of squares.

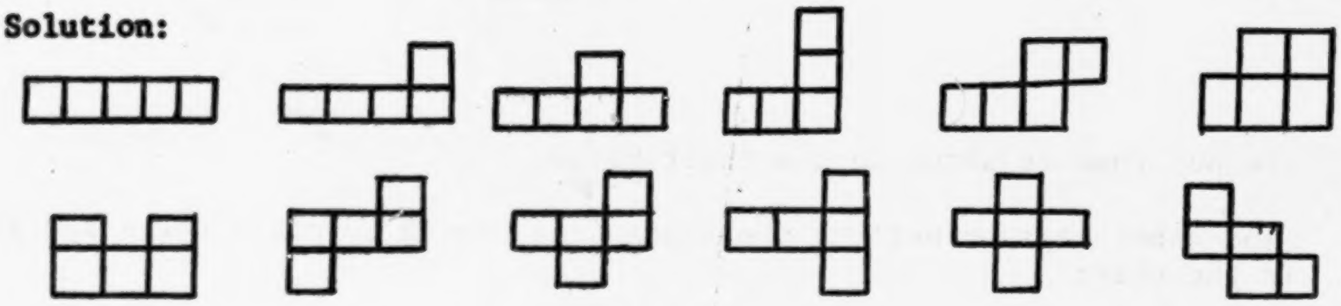
a. How many different (distinct) ways can you make a connected set of 4 squares?

Note that  and  are the same connected set of 4 squares. The set has been moved to different positions.

Solution: 

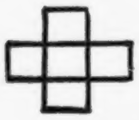
b. How many different (distinct) connected sets of 5 squares can you form?

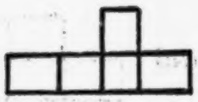
Solution:

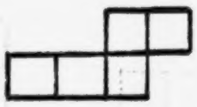


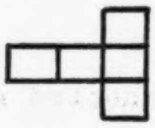
NOTE: Contrasting colored graph paper of 1" x 1" x 1" squares could be furnished to pupils for problems "a" and "b".

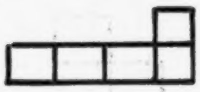
c. Which of the patterns below can be folded into an open box?

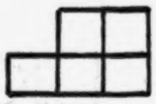
A


B


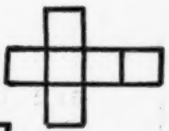
C


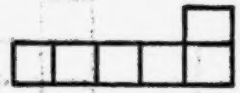
D


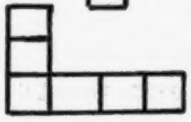
E


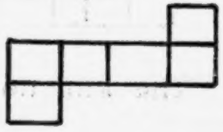
F


d. Will any of these shapes fold into a cube? Which ones?

A


B


C


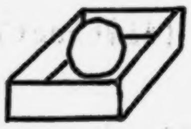
D


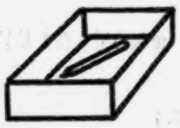
Solution: A, D

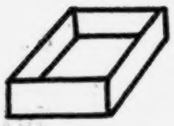
NOTE: Pupils could use paper patterns to fold for problems c and d.

7. Here are 3 boxes and an orange and a pencil.

Sam put them as shown


Box A


Box B


Box C

Jim put them as shown in the chart below.

Find other ways of putting the orange and pencil in the 3 boxes and fill in the chart

	Box A	Box B	Box C	Equation
Sam	1	1	0	1 + 1 + 0 = 2
Jim	2	0	0	2 + 0 + 0 = 2

ALGEBRA FOR GRADE 8

An algebra course for grade 8 should appear as a natural outgrowth of the concepts enunciated in Strands 1-9. For this reason we need primarily to list topics and prescribe proficiencies which should be achieved. In these brief remarks we try to point out ways of looking at traditional topics in algebra which reflect our primary concern with basic mathematical concepts.

One of the immediate goals for a first course in algebra is a study of the important properties of the real number system which pertain to and make possible the solution of equations and inequalities. While this is essentially an extension of Strand 1, Numbers and Operations, solutions of these systems must be approached through the avenues of functions and be related to geometry by graphing on the real number line and in the real number plane. The class of equations considered must find their motivations within meaningful applications.

It is appropriate to review the important field properties of the rational numbers. These properties were listed in a table on page 10. A useful discussion can be centered around the set of numbers $\{a + b\sqrt{2}:\ a \text{ and } b \text{ rational}\}$. This set of numbers is closed under addition and multiplication and satisfies all of these field properties.

The set of functions which are studied in this first course in algebra are almost exclusively real valued functions. We can now give a careful definition of a real valued function from the point of view of ordered pairs of real numbers and identify the domain and range of a function and stress the mapping concept associated with functions. Processes of constructing new functions from old ones should be identified. These processes should be linked with techniques of graphing. The important operations are:

Addition: If f and g are functions, the function $(f + g)$ is defined by $(f + g)(x) = f(x) + g(x)$,

Multiplication: If f and g are functions, the function $(f \cdot g)$ is defined by $(f \cdot g)(x) = f(x) \cdot g(x)$, and

Composition: If f and g are functions, the function $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$.

These definitions do of course require some restrictions on the domain and range of the functions. For addition and multiplication the functions f and g must have a common domain. The domain of $(f + g)$ and $(f \cdot g)$ is the intersection of the domains of f and g. For the composition $(f \circ g)$ to have meaning, the range of g must be contained in the domain of f. However, for this first introduction, mathematical correctness can be preserved with little loss in utility if we consider only functions whose domain is the set of all real numbers, or, at worst, the set of positive real numbers. Sufficient examples should be given to show that composition is not commutative.

Example: If $f: x \rightarrow x^2$ and $g: x \rightarrow x + 2$ then
$f \circ g: x \rightarrow (x + 2)^2$ while $g \circ f: x \rightarrow x^2 + 2$.

Thus: $f \circ g \neq g \circ f$.

When composition as an operation on functions has been mentioned, the notion of an inverse for a function should be introduced.

Inverse: If f is a function, a function g is called the inverse of f if $g \circ f$ is the identity function, i.e.: $(g \circ f)(x) = x$ for all x in the domain of f.

The inverse function must of course be clearly differentiated from the reciprocal function. The simple relation between the graph of a function and the graph of its inverse should be emphasized.

Example: If f is the function $f(x) = x + 2$, then the inverse of f is the function g defined by $g(x) = x-2$, while the reciprocal function is the function h defined by $h(x) = \frac{1}{x + 2}$. Once more, if g is the inverse function for f, then $f \circ g$ is the identity function. If h is the reciprocal function for f, then $f \cdot h$ is the constant function 1.

We recommend that a first study be restricted to the class of linear functions. These are functions of the form

$$f: x \rightarrow ax + b \qquad \text{or} \qquad f(x) = ax + b.$$

Many applications of mathematics lead to linear functions. Problems involving rate, time and distance are in this category.

Many functions can be reexpressed as (linear) functions in this compact form. Indeed the traditional problems of collecting "likes with likes" are included here.

Example. The following functions are all linear:

$$f(x) = 2x + 5 - x$$
$$g(x) = 6 - \frac{7}{2} + \frac{3}{4}x$$
$$h(x) = (x-1)(x + 1) - x^2 + 2x.$$

Justification for the steps in the simplification used here should be discussed and some exercises designed to achieve proficiency in these techniques should be included.

It is important to note that except for the constant functions (the case $a = 0$) these functions have inverses which are also linear. Linear functions also have other nice properties; they are closed under the operations of addition and composition--but not under multiplication. In this way, the mathematical system of linear functions has many properties of the integers under addition.

It is important to relate these functions to their graphs in the number plane. We recommend giving an informal demonstration, based on the geometric concepts of similarity already introduced in the K-8 program, that the graph of a linear function is a straight line, and (with the obvious exceptions) conversely. The usual trouble with lines which are parallel to the y-axis arises, but from such a discussion we can introduce relations and linear systems of equations and inequalities in two indeterminates.

We recommend solving linear inequalities graphically as well as algebraically and providing a display of the results both on the number line and in the number plane.

Example: For what values of x is $-2 < 2x + 1 < 2$?

Graphic solution: Graph the function $f: x \rightarrow 2x + 1$

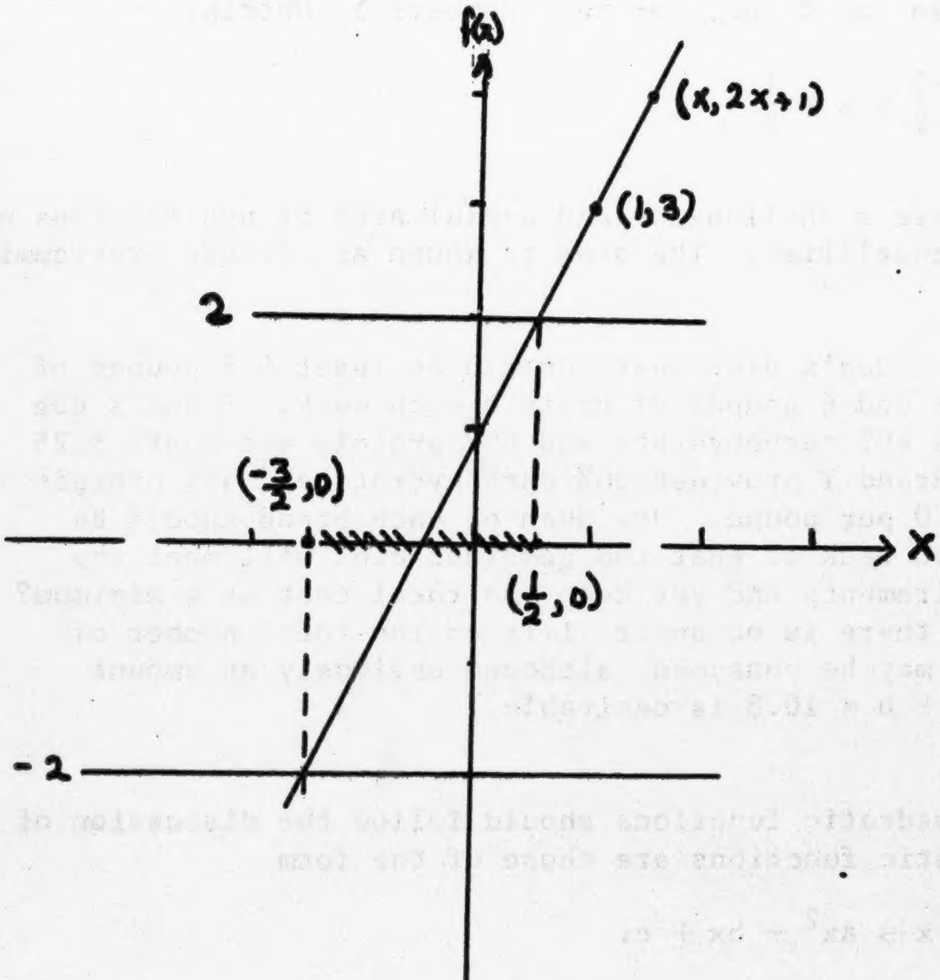


Figure 1

Identify the interval (-2,2) in the range of the function. Find the corresponding interval in the domain. Conclude (region shown with hash marks) that if $\frac{-3}{2} < x < \frac{1}{2}$ then $-2 < 2x + 1 < 2$.

Algebraic solution:

$$-2 < 2x + 1 < 2$$

1. Add (-1) to each term of the inequality. (This is possible because of the rule, if $a < b$ then $a + c < b + c$, for real numbers.) Obtain:

$$-3 < 2x < 1$$

2. Multiply each term of the inequality above by $\frac{1}{2}$. (This is possible because of the rule, if $a < b$ and $c > 0$ then $ac < bc$, for real numbers.) Obtain:

$$\frac{-3}{2} < x < \frac{1}{2} \cdot$$

We mention here a challenging and useful area of applications of linear equalities and inequalities. The area is known as linear programming. Here is an example.

> Suppose that a dog's diet must contain at least 4.8 pounds of carbohydrates and 6 pounds of protein each week. Brand X dog food provides 40% carbohydrate and 60% protein and costs \$.25 per pound. Brand Y provides 60% carbohydrate and 40% protein and costs \$.20 per pound. How much of each brand should be purchased each week so that the combined diet will meet the minimum requirements and yet keep the total cost at a minimum? (Assume that there is no upper limit on the total number of pounds which may be consumed, although obviously an amount close to 4.8 + 6 = 10.8 is desirable.)

A study of quadratic functions should follow the discussion of linear functions. Quadratic functions are those of the form

$$f: \; x \rightarrow ax^2 + bx + c.$$

By permitting $a = 0$, we can include the linear functions in this class. However we recommend the convention of using the term "quadratic functions" to mean that $a \neq 0$. Once again, to recognize which functions are quadratic, or to have facility in operating with them, we need considerable practice in the manipulation of these expression in x. It is important to study these functions as affected by the parameters a and c. Thus compare $g(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ and $f(x) = ax^2 + bx + c = a \cdot g(x)$. Compare $h(x) = x^2$ and $k(x) = x^2 + c = h(x) + c$.

We shall of course require the discriminant and the quadratic formula to determine the real zeros of quadratic functions. We recommend that heavy emphasis be placed on the process of "completing the square" which turns out to be more useful than the quadratic formula itself. For example, given both the functions $f(x) = x^2 + 2x + 3$ and $g(x) = (x + 1)^2 + 2$, we can easily prove that $f(x) = g(x)$ for all x. However it is more important to be able to transform the form of the function $x^2 + 2x + 3$ into the form $(x + 1)^2 + 2$, and to know that such a procedure will be helpful. Through these techniques every student should know that any quadratic function is essentially like the squaring function $x \rightarrow x^2$.

For the class of quadratic and linear functions we should study the operations of addition, multiplication, composition and the existence of inverses of functions. The comparison of the behavior of all quadratic functions with the function $x \rightarrow x^2$ should be related to the effect of a translation of coordinates.

A host of interesting and important applications can be approached by considering various isoperimetric problems involving rectangles.

> Example: A boy has some money to buy fencing to build a pen for his pet rabbit. He has decided to build a rectangular pen since this simplifies construction problems. He may build it either within the center of the yard, or along a two foot stretch of the neighbors' fence. If he builds in the center of the yard he may use arbitrary dimensions (except that of course he can't exceed the length of fencing he may buy)! If he builds along the neighbors' fence he can save that two foot stretch of fencing, but in this case he must build a pen that is only two feet wide. Where, and in what shape should he build the pen to make most effective use of his fencing; that is, how can he get the biggest area for his money?

Discussion of this problem will include the extreme value problems of a little and a lot of fencing. It will also include what shape he should build if he builds in the yard. We should conclude that of all rectangles with a fixed perimeter, the square has the largest area. This discussion will include the corollary that the geometric mean is smaller that the arithmetic mean: $\sqrt{xy} < \frac{x+y}{2}$ by comparing rectangles of size x by y and the square of side $\frac{x+y}{2}$.

In connection with quadratic functions it would be natural to include a brief introduction to complex numbers. If we denote a solution of $x^2 = -1$ by i, we may then study the set of numbers $\{a + bi\colon\ a, b \text{ real}\}$ under the usual definitions for addition and multiplication. It is easy to show that these numbers, just like the set $\{a + b\sqrt{2}\colon\ a, b \text{ rational}\}$ satisfy the field properties listed on page 10, except for the ordering properties. If the complex numbers are introduced, it could be shown that every equation of the form $ax^2 + bx + c$ where now a, b, and c may be complex numbers, has a solution in the field of complex numbers. The geometric representation of complex numbers as points in the complex plane might be included as well.

In this first course in algebra we do not believe that the full scope and power of the class of all polynomial functions need be stressed. In particular we do not recommend treating polynomials as expressions or forms in an indeterminate. All discusssions should be kept as concrete as possible by discussing functions.

A class of functions we do want to introduce are those involving exponents and radicals. In developing the traditional sequence of functions of the type

$$f:\ x \rightarrow x^n \quad \text{where } n \text{ is a positive integer}$$

the essential difference of the even and odd cases should be examined. It is important to study the behavior of these functions as they depend on the parameter n. Of course the standard rules for operating with exponents must be covered.

Next we study functions of the form

$$f:\ x \rightarrow x^{1/n}$$

There are a variety of motivations which can and should be used to investigate these two classes of functions. One that should be included is that of seeking an inverse for all or part of the function $f(x) = x^n$ where n is a positive integer. In particular we follow the convention that $x^{1/2}$ denotes the positive number, if any, whose square is x. Thus $(y^2)^{1/2} = |y|$

In studying exponential functions it is important that some time be spent in computing with rational values that "work out" nicely. In this way the construction and use of a table of the integral powers of 2 is a useful learning experience.

It is but a short step to fractional exponents; that is, functions of the type

$$h:\ x \rightarrow x^{p/q} \quad \text{where } \frac{p}{q} \text{ is a rational number.}$$

The law, valid for positive numbers x, which states that $(x^p)^{1/q} = (x^{1/q})^p$ may also be viewed as stating a result about the composition of the functions

$$f:\ x \rightarrow x^p \quad \text{and} \quad g:\ x \rightarrow x^{1/q} \quad (x \geq 0).$$

This result is that $h = f \circ g = g \circ f$.

Finally, negative exponents should be introduced.

There are many subtleties in the treatment of exponents. One concerns the standard laws:

$$a^r a^s = a^{r+s}$$

$$(a^r)^s = a^{rs}$$

$$a^r b^r = (ab)^r.$$

These hold for certain combinations of a,b, r and s, but not for others. They all do hold when a and b are positive. On the other hand we contrast the standard paradox

$$-1 = (-1)^1 = (-1)^{2/2} = ((-1)^2)^{1/2} = 1^{1/2} = 1$$

with

$$-1 = (-1)^1 = (-1)^{3/3} = ((-1)^3)^{1/3} = (-1)^{1/3} = -1.$$

This subtlety must be explored!

Another subtlety concerns inequalities and order. Note that $\sqrt{x} < x$ if $x > 1$, but $x < \sqrt{x}$ if $x < 1$. For what combinations of x and rational r is $x^r > x$? On the other hand, the exponential functions are monotonic; that is, if $x < y$ then $x^r < y^r$ for positive exponents r.

It is desirable that the student have an intuitive grasp of the continuity of the functions.

$$f:\ x \rightarrow x^n \quad \text{and} \quad g:\ x \rightarrow x^{1/n} \text{ for integral } n.$$

Sufficient examples should be provided so that each pupil will get a feel for the fact that, for example, if a and b are close together, then a^n and b^n are close together, and conversely. Each pupil should know that each positive real number has an n^{th} root, and should know some techniques for finding an approximation. Such familiarity is important before tackling such problems as the meaning of $2^{\sqrt{2}}$. Irrational exponents should be discussed, but in no great detail. We do recommend that a basis for later work with logarithms be laid. An excellent way to do this is to construct a table of the powers of 10 using estimates like $10^3 \sim 2^{10}$ and so $2 \sim 10^{.3}$. A complete treatment of this approach is given in "Goals for School Mathematics", the Report of the Cambridge Conference on School Mathematics, (pages 73-76), Houghton Mifflin Company, Boston, Massachusetts, 1963.

The study of rational functions of the form

$$r:\ x \rightarrow \frac{2x^2 + 3x + 5}{x - 3}$$

or more generally of the form

$$s:\ x \rightarrow \frac{f(x)}{g(x)}$$

where f and g are polynomial functions, should be undertaken from the point of view of what properties the function inherits from those possessed by f and g. Many of the traditional algebraic skills are acquired in drill in simplifying functions of this type. The similarity with rational numbers should be observed in carrying out these manipulations.

The crucial subtlety in dealing with rational functions is the determination of the domain of the function. We want to be correct, but not pedantic. We want students to learn to take care in making combinations, but we do not want artificiality to pass for mathematical sophistication. For example, suppose we are given two functions, f and g.

$$f:\ x \longrightarrow \frac{x^2}{x-1} \qquad \text{and} \qquad g:\ x \rightarrow \frac{1}{x-1}\ .$$

For each of these functions we exclude (1) from the domain. But what of the function f-g?

$$f - g:\ x \rightarrow \frac{x^2}{x-1} - \frac{1}{x-1}\ .$$

We see that $\frac{x^2}{x-1} - \frac{1}{x-1} = \frac{x^2-1}{x-1}$ for all $x \neq 1$.

The last expression is of course not reduced to "simplest terms". If we do this we obtain $x + 1$. It is true that $f\text{-}g:\ x \rightarrow x + 1$ for all $x \neq 1$, but we have no mathematical right to claim that $(f\text{-}g)\ (1) = 0$. The question we are really asking is, for what x is

$$\frac{x^2-1}{x-1} = x + 1?$$

This is probably too subtle a question to be considered in a first course in algebra. Certainly it should not be made an issue, but it should be pointed out that "identities" like the one above hold only for numbers in the domain of the two functions.

There should be space and time in this algebra syllabus for the inclusion of some topics from elementary number theory. This subject can provide an appropriate setting for many principles of logical thinking. A non trivial exercise in logical thinking and number theoretic problems is to show that to determine whether an integer, say 1107, is a prime, we need only check as possible factors those primes less than or equal to its square root, in this case, the primes up to and including 31.

GOALS

The following statement of goals for the student is in general terms; we refer to the main body of this report for indications of depth and extent of coverage. We believe that the following objectives can be stated and tested in behavioral terms.

1. Numbers and Operations

To use effectively the fundamental operations of arithmetic, computing with fractions and with decimals; to understand and utilize the properties (commutative property, etc.) of the operations, and the properties of order and absolute value; and to understand the structure of the several number systems and the special properties of each.

To read and understand mathematical sentences involving operations, exponents, and letters, and to formulate and use such sentences in the analysis of mathematical problems.

2. Geometry

To recognize and use common geometric concepts and configurations; to utilize compass and straight edge for simple constructions; and to understand and to construct simple deductive proofs.

To use the elementary quantitative geometric notions, such as measure of angle, area and volume; to utilize the concepts of similarity and congruence in applications such as plans and maps; and to utilize the coordinate plane.

3. Measurement

To make measurements; to understand the notion of unit of measurement, and to use and interpret various units; to understand the degree of accuracy of an approximate measurement; to estimate measurements and the results of simple calculations involving measurements; and to conceive and use forms of measurement as functions.

4. Applications

To analyze concrete problems by using an appropriate mathematical model; to employ graphs, scale drawings, sentences, formulae, computations and reasoning in studying the mathematics of such a model; to interpret mathematical consequences in concrete terms; and to examine the concrete results of such an analysis in terms of reasonableness and accuracy.

5. Statistics and Probability

To construct and read ordinary graphs.

To collect and organize data by means of graphs and tables; to interpret data using concepts describing central tendency, such as mean, median and mode; and to understand statistical variance as a measure of central tendency.

To understand, at a simple level, the idea of sampling, and to interpret and predict from data samples.

To understand rudimentary notions of probability theory and of chance events.

6. Sets

To understand and use routinely the basic set concepts, notations, and operations.

7. Functions and Graphs

To use the coordinate plane to display relations and to organize data; to recognize and utilize the concept of function, or functional relation; and to use functions and the usual functional notation in analysis and problem solving.

8. Logical Thinking

To understand, to appreciate, and to use precise statements; to understand and use correctly the simple logical connectives such as: "and", "if-then", etc.; to distinguish, conceptually and in operations, between the "for some" and "for all" quantifiers; and to follow and to construct simple deductive arguments.

9. Problem Solving

To devise and apply strategies for analysis and solution of problems, and to use estimation and approximation to verify the reasonableness of the outcome.

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